Universidade de Aveiro Departamento de Matemática
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Aurineide Castro Fonseca

Teoria de Funções para Operadores Fracionários de Dirac

Function Theory for Fractional Dirac Operators

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Dissertação apresentada à Universidade de Aveiro e à Universidade do Minho para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática e Aplicações, realizada sob a orientação científica da Prof.a Dr.a Paula Cerejeiras, Professora Associada do Departamento de Matemática da Universidade de Aveiro.

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Palavras-chave

Álgebras de Clifford; Álgebras ternárias; Polinómios monogénicos fracionários; Decomposição de Fischer; Análise fracionária de Clifford; Operador de Dirac fracionário; Operadores de derivação generalizada de Gelfond-Leontiev; Teorema de Cauchy-Kovalevskaya; Transformada fracionária de Fourier;

Resumo

Nesta tese são estudadas algumas das ferramentas básicas da teoria de funções fracionárias em dimensões superiores, por meio de uma correspondência fracionada com as relações de Weyl através de operadores de diferenciação generalizada de Gelfond-Leontiev (GL). Estabelecemos uma decomposição de Fischer e apresentamos um algoritmo para a construção de polinómios monogénicos homogéneos fracionários de grau arbitrário. Também descrevemos a análise generalizada de Clifford fracionada na configuração ternária e daremos uma descrição completa, algébrica e analítica, dos espaços de funções monogénicas neste sentido, sua decomposição Fischer análoga e concluindo com uma descrição da base para os espaços de polinómios homogéneos monogénicos fracionários que surgem neste caso e um algoritmo explícito para a construção desta base.

Esta teoria também inclui o teorema de extensão de Cauchy-Kovalevskaya (CK) na configuração fracionária generalizada usando os operadores de derivação GL e são apresentados os módulos de Hilbert com núcleo reprodutivo que surgem a partir das potências monogénicas construídas através do teorema CK. Além disso, damos as transformações semelhantes a Fourier ligadas a operadores de derivados fracionados em relação aos operadores GL de diferenciação generalizada e suas propriedades.

Keywords

Clifford algebras; Ternary algebras; Fractional monogenic polynomials; Fischer decomposition; Fractional Clifford analysis; Fractional Dirac operator; operators of generalized differentiaton of Gelfond-Leontiev; Cauchy-Kovalevskaya theorem; Fractional Fourier transform

Abstract

This thesis studies the basic tools of a fractional function theory in higher dimensions by means of a fractional correspondence to the Weyl relations via Gelfond-Leontiev operators of generalized differentiation. A Fischer decomposition is established and we give an algorithm for the construction of monogenic homogeneous polynomials of arbitrary degree. We also describes the generalized fractional Clifford analysis in the ternary setting and we will give a complete algebraic and analytic description of the spaces of monogenic functions in this sense, their analogous Fischer decomposition, concluding with a description of the basis of the space of fractional homogeneous monogenic polynomials that arise in this case and an explicit algorithm for the construction of this basis.
This theory also includes the Cauchy-Kovalevskaya (CK) extension theorem in the generalized fractional setting by using the GL derivative operators and it is presented the reproducing kernel Hilbert modules that arise from the monogenic formal powers constructed via the CK theorem. In addition, we give the Fourier-like transforms linked to fractional derivative operators with respect to Gelfond-Leontiev operators of generalized differentiation and its properties.

Contents

Introduction

The function theory for null-solutions of the Dirac operator defines the so-called classic Clifford analysis, or the theory of monogenic functions, that is, solutions of the mass-less Dirac operator. The motivation for studying this subject is due the fact that Clifford analysis, among other things, represents one of the most remarkable field of modern mathematics with a higher dimensional generalization of Complex Analysis. These null-solutions, also called monogenic functions, replace the notion of a holomorphic function, i.e. the Dirac operator D on $\mathbb{R}^d$, a conformally invariant first-order differential operator, plays the same role in classical Clifford analysis as the Cauchy-Riemann operator ∂_z does in the complex analysis.

While the monogenic functions can very well describe problems of a particle with internal $SU(2)$-symmetries, higher order symmetries are beyond the reach of this theory. Although many modifications were suggested over the years (such as Yang-Mills theory) they could not address the principal problem, the need of a n-fold factorization of the d'Alembert operator.

Over the last decades F. Sommen and his collaborators developed a method for establishing a higher dimension function theory based on the construction of a so-called Howe dual pair consisting of a Super-Lie algebra (usually $\mathfrak{osp}(1|2)$) and a Spinor space. This Super-Lie algebra $\mathfrak{osp}(1|2)$ is then generated by three operators: the Dirac operator, the vector variable operator, and the so-called Euler operator or radial derivative; the latter operator arises as the anti-commutator between the Dirac operator and the vector variable operator and has as eigenspaces the space of homogeneous polynomials.

In the setting of Dirac operators with fractional derivatives, this approach does not work as well as in the classic case since, in general, they do not allow a construction of a Howe dual pair. Hereby, the principal problem is not the invariance under a fractional spin group, but the construction of a Super-Lie-algebra $\mathfrak{osp}(1|2)$ for general fractional Dirac operators. This requires the application of a new approach and new methods, in particular the establishment of fractional Sommen-Weyl relations.

The traditional Fischer decomposition in harmonic analysis yields an orthogonal decomposition of the space of homogeneous polynomials of fixed degree of homogeneity in terms of spaces of harmonic homogeneous polynomials. In classical continuous Clifford analysis one obtains a refinement yielding an orthogonal decomposition with respect to the so-called Fischer inner product of homogeneous polynomials in terms of spaces of monogenic polynomials, i.e., null solutions of the Dirac operator (see [31]). Generalizations of the Fischer decomposition in other frameworks can be found, for example, in [11, 13, 36, 5, 26] and mainly based on the establishment of raising and lowering operators acting on a ground state.

As we will see in this thesis, the fractional Dirac operators based on Gelfond-Leontiev (GL) operators of generalized differentiation still allow for a Fischer decomposition and while not providing an $\mathfrak{osp}(1|2)$ algebra realization they still permit the explicit construction of the basis of fractional homogeneous monogenic polynomials acting on a ground state.

This dissertation is organized in six chapters. In chapter [1] we will recall some important facts about Clifford algebras and monogenic functions. We will also give some basic concepts on reproducing kernel Hilbert modules in the case of Clifford-valued functions.

In chapter [2] we will see the definition of Riemann-Liouville and Caputo fractional derivatives and some of its properties, the concept of the GL derivative and integral operators and it is given the eigenvalues and eigenfunctions of the fractional Laplace operator associated to GL derivative operator.

It is well known that the concept of fractional derivatives is not new. In fact, its beginning was in a letter by Leibnitz to L'Hospital, in 1695, which had a question about the possibility defining $\frac{d^n}{dx^n}$ for non-integral values of n. In reply, L'Hospital wondered $\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}$ which Leibnitz responded prophetically: *"It leads to a paradox, from which one day useful consequences will be drawn"*.

The interest in generalized fractional calculus has increased substantially in more recent times. This increase is due, in part, to some of its interesting special cases and its applications to different topics of analysis. The basic operators of this calculus, the so-called generalized (multiple) operators of fractional integration are defined by means of convolutional type single integrals involving the Meijer functions $G^{m,0}_{m,m}$ as kernels. The classical Riemann-Liouville fractional integrals are a particular case of these general fractional integrals.

In a more general case, the $H-$functions of Fox appear as kernel-functions of those convolutional type single integrals. These same generalized fractional integral operators can be considered also as compositions of an arbitrary number of commuting Erdélyi-Kober fractional integrals written as repeated integrals without any use of special functions. Since such compositions arise quite often in several problems and branches of different areas of applied mathematics this relationship proves to be the key to applications of the fractional calculus. From the long list of applications we have: differential and integral equations, operational calculi, convolutional calculi, integral transforms, univalent functions and special functions, among many more.

In chapter [3] we will give a detailed description of the ternary algebras. This definition is very important since, for instance to obtain a Dirac operator D such that $D^3 = \Delta$ a real Clifford algebra is not enough and we need to define a so-called generalized Clifford algebra. In addition we will obtain the symmetries for the case of the dimension 2 of these algebras.

In chapter [4] we will present the basic tools of fractional function theory in higher dimensions, in the classic Clifford algebra and ternary case, by means of a fractional correspondence to the Weyl relations via GL operators of generalized differentiation. These operators GL were studied in the 80's and 90's and allow particular realizations in form of the classical Caputo and Riemann-Liouville fractional derivatives.

Instead of constructing a Super-Lie algebra associated with our structures we will construct a so-called Fischer pair base and introduce an inner product onto the space of homogeneous polynomials with values in the generalized Clifford algebra. It is very clear that this cannot be the same inner product as in the case of the classic Clifford algebra, but it can be produced by a clever introduction of a conjugation operator. While this allows for the construction of a Fischer decomposition we encounter an additional problem. Due to the lack of the $\mathfrak{osp}(1|2)$-property, in particular the lack of a suitable Euler operator, we cannot use the standard ansatz for the monogenic projection, i.e. the projection of an arbitrary homogeneous polynomials into the space of monogenic homogeneous polynomials. To overcome this problem we are going to show that this projection can be recast as a linear system with coefficients in the generalized Clifford algebra. Although linear algebra with respect to Clifford-valued matrices is a difficult topic we can show that the resulting system is solvable in our case.

We will see that ternary Clifford algebra will provide a cubic factorization of the Laplacian and we will analyze the associated function theory. We will use a computer algebra system to compute the coefficients of the monogenic homogeneous polynomials that form the basis of the space of fractional homogeneous monogenic polynomials that arise in this case.

In chapter [5] we will give the Cauchy-Kovalevskaya (CK) extension theorem in the generalized fractional setting by using the GL derivative operators. We will also present the reproducing kernel Hilbert modules that arise from the monogenic formal powers constructed via the CK theorem.

Finally, in chapter [6] we will present the building blocks of a theory for Fourier-like transforms linked to fractional derivative operators with respect to GL operators of generalized differentiation.

Chapter 1

Preliminaries

In this chapter we present a short resume of the preliminary concepts necessary for the good understanding of this thesis. First, we make a brief description of Clifford algebras, after which we described the (now-a-days) classical Clifford analysis, with emphasis on the main techniques and methods in this field. We finalize this section with the concept of reproducing kernel Hilbert modules.

1.1 Clifford Algebras

Clifford algebras were introduced by William Kingdon Clifford [41] in 1878 as a generalisation to higher dimensions of complex numbers. These algebras have a deep connection to the concept of *geometric algebras* (that is, unital associative algebras generated by a vector space endowed with a quadratic form). Moreover, they are connected with the theory of bilinear forms and orthogonal transformations which makes them important tools for several fields, like mathematical physics, differential geometry, or digital image processing.

For further details about Clifford algebras and basic concepts of the associated function theory we refer the interested reader to [9, 21, 31].

1.1.1 Real Clifford Algebras

Let X be a finite dimensional real vector space with $\dim X = d$ and let $\mathcal{B}$ be a non-degenerate symmetric bilinear form on X, i.e. $\mathcal{B} : X \times X \to \mathbb{R}$ satisfies to

(B1) linearity on the first component: for all $\lambda \in \mathbb{R}$, $v, v', w \in X$ we have

$$\mathcal{B}(\lambda v + v', w) = \lambda \mathcal{B}(v, w) + \mathcal{B}(v', w),$$

(B2) Symmetry: for all $v, w \in X$ it holds

$$\mathcal{B}(v, w) = \mathcal{B}(w, v).$$

(B3) Non-degeneracy: for each non-zero $v \in X$ there exists a $w \in X$ such that

$$\mathcal{B}(v, w) \neq 0.$$

From properties **(B1)** and **(B2)** one obtains the bi-linearity of $\mathcal{B}$, and the pair $(X, \mathcal{B})$ is said to be a non-degenerate $d-$dimensional real orthogonal space.

We also have that if $A = (a_{ij})$ is the $d \times d$ real matrix of $\mathcal{B}$ relative to a arbitrary ordered basis for X then the symmetry and non-degeneracy properties of $\mathcal{B}$ imply that A is symmetric and non-singular. Thus, there exists an ordered basis $e = \{e_1, \cdots, e_d\}$ for X and a pair $(p, q) \in \mathbb{N}_0^2$, with $p + q = d$, for which it holds

(1) $\mathcal{B}(e_i, e_i) = 1, \quad i = 1, \cdots, p;$

(2) $\mathcal{B}(e_i, e_i) = -1, \quad i = p + 1, \cdots, p + q = d;$

(3) $\mathcal{B}(e_i, e_j) = 0, \quad i \neq j.$

We say that such a basis e is an orthonormal basis for X which diagonalizes the bilinear form $\mathcal{B}$. Moreover, the Sylvester Theorem ensures the independence of the pair (p, q) with respect to the orthonormal basis e. In particular, the non-degenerated orthogonal space $\mathbb{R}^{p,q} := (\mathbb{R}^d, \mathcal{B})$ with $p + q = d$ and

$$\mathcal{B}(x, y) := \sum_{j=1}^{p} x_j y_j - \sum_{j=p+1}^{p+q} x_j y_j$$

is called the *real vectorial space of signature* (p, q) and it is always possible to construct an isomorphism between the real orthogonal space $(X, \mathcal{B})$ and $\mathbb{R}^{p,q}$.

We now introduce the definition of a Clifford algebra associated to a given non-degenerate d-dimensional real orthogonal space $(X, \mathcal{B})$ with signature (p, q).

Definition 1.1 (Real Clifford algebra)**.** *A real associative algebra $\mathcal{A}$ with identity* 1 *which satisfies*

(a) *$\mathcal{A}$ contains copies of $\mathbb{R}$ and X as linear subspaces;*

(b) *For all $v \in X$, $v^2 = \mathcal{B}(v, v)$*

(c) *$\mathcal{A}$ is generated as a ring by the copies of $\mathbb{R}$ and X,*

is said to be a real Clifford algebra for $(X, \mathcal{B})$ and it is denoted by $\mathcal{A} = C(X)$.

Moreover, if $\dim C(X) = 2^d$ then $C(X)$ is called a *universal Clifford algebra* for $(X, \mathcal{B})$. The isomorphism between the orthogonal spaces $(X, \mathcal{B})$ and $\mathbb{R}^{p,q}$ implies that we may restrict ourselves to the study of $\mathcal{A} = C(\mathbb{R}^{p,q})$.

If, in particular, $\mathcal{A} = C(\mathbb{R}^{p,q})$ is a universal Clifford algebra then we signalise this fact by writing $\mathcal{A} := \mathbb{R}_{p,q}$.

We now present some elementary examples.

Example 1.1. *The complex algebra $\mathbb{C}$ can be identified with the universal Clifford algebra $\mathbb{R}_{0,1}$, generated by e_1 such that $e_1^2 = -1$. We write then $\mathbb{R}_{0,1} \simeq \mathbb{C}$.*

Example 1.2. *The algebra of real quaternions $\mathbb{H} = Span_{\mathbb{R}}\{1, i, j, k\}$ can be identified with $\mathbb{R}_{0,2}$, the universal Clifford algebra where $e_1^2 = e_2^2 = -1$, via the identification*

$$1 \to 1, \qquad i \to e_1, \qquad j \to e_2, \qquad k \to e_1 e_2;$$

however, k no longer represents a vector. A second possible identification is obtained by identifying the elements of $\mathbb{H}$ with a subalgebra of $\mathbb{R}_{0,3}$ by

$$1 \to 1, \qquad i \to e_1 e_2, \qquad j \to e_1 e_3, \qquad k \to e_2 e_3$$

but again, vectors of $\mathbb{H}$ are identified with bi-vectors (oriented areas) of $\mathbb{R}_{0,3}$.

$\mathbb{H}$ is an example of a non universal Clifford algebra since it has dimension $2^2 = 4$ while being generated by a 3-dimensional vector space.

Finally, we look into the problem of construction of a basis for the universal real Clifford algebra $\mathbb{R}_{p,q}$. Given an orthonormal basis $e = \{e_1, \cdots, e_d\}$ we have

$$e_i^2 = \begin{cases} 1, & \text{if } i = 1, \cdots, p; \\ -1, & \text{if } i = p+1, \cdots, p+q. \end{cases}$$

Moreover, from

$$\mathcal{B}(e_i + e_j, e_i + e_j) = (e_i + e_j, e_i + e_j)^2 = e_i^2 + e_i e_j + e_j e_i + e_j^2$$

and

$$\mathcal{B}(e_i + e_j, e_i + e_j) = \mathcal{B}(e_i, e_i) + \mathcal{B}(e_j, e_j) + 2\mathcal{B}(e_i, e_j) = e_i^2 + 2\mathcal{B}(e_j, e_j) + e_j^2,$$

we get

$$e_i e_j + e_j e_i = 2\mathcal{B}(e_i, e_j) = 0, \text{ for all } i \neq j.$$

It follows from these considerations that a basis for $\mathbb{R}_{p,q}$ is given by the ordered products of different elements of e, that is by

$$e_\emptyset := 1, \quad e_A := e_{l_1} e_{l_2} \dots e_{l_r}, \quad \text{where } A = \{l_1, l_2, \dots, l_r\} \text{ with } 1 \leq l_1 < \dots < l_r \leq d.$$

Hence, an arbitrary element $a \in \mathbb{R}_{p,q}$ can be written as

$$a = \sum_A a_A \, e_A, \quad a_A \in \mathbb{R}.$$

1.1.2 $k-$vectors

The linear subspace of $\mathbb{R}_{p,q}$ spanned by the elements of the basis e_A for which $|A| := \# A = k$ is denoted by

$$\mathbb{R}^k_{p,q} = \{a \in \mathbb{R}_{p,q} : a = \sum_{|A|=k} a_A e_A\}, \quad k = 0, 1, \cdots, d,$$

and its elements $a = \sum_{|A|=k} a_A e_A$ are called k-vectors.

Example 1.3. *We give some particular cases.*

(a) $\mathbb{R}^0_{p,q}$ *is the 1-dimensional subspace of* $0-$*vectors, or scalars.*

(b) $\mathbb{R}^1_{p,q}$ *is the d-dimensional subspace of* $1-$*vectors, or vectors for short, and it is usually identified with* $\mathbb{R}^{p,q}$.

(c) $\mathbb{R}^2_{p,q}$ *is the* $\binom{d}{2}$*-dimensional subspace of* $2-$*vectors, or bi-vectors, and its elements are called bivectors.*

(d) $\mathbb{R}^0_{p,q} \oplus \mathbb{R}^1_{p,q}$ *is the* $d+1$*-dimensional vector space generated by adding scalars and vectors, and its elements are called paravectors. By convention, we write a paravector as*

$$x = x_0 + \underline{x},$$

where $x_0 \in \mathbb{R}^0_{p,q}$ *and* $\underline{x} \in \mathbb{R}^1_{p,q}$.

The mapping $[\,\cdot\,]_k : \mathbb{R}_{p,q} \to \mathbb{R}^k_{p,q}$ denotes the projection operator of $\mathbb{R}_{p,q}$ onto $\mathbb{R}^k_{p,q}$. Thus, the following decomposition holds

$$\mathbb{R}_{p,q} = \mathbb{R}^0_{p,q} \oplus \mathbb{R}^1_{p,q} \oplus \dots \oplus \mathbb{R}^d_{p,q},$$

meaning that an element $a \in \mathbb{R}_{p,q}$ can be rewritten as $a = \sum_{k=0}^d [a]_k$.

1.1.3 Conjugation and the Orthogonal Group

A key point in Clifford algebras is the fact that an arbitrary product of two vectors can be decomposed into the sum of a symmetric and an anti-symmetric parts, thus emphasising the inherent $SO(2)$ nature of these algebras.

In fact, for all $x, y \in \mathbb{R}^1_{p,q}$ we obtain that their product xy can be split up into

$$xy = \frac{xy + yx}{2} + \frac{xy - yx}{2} := x \cdot y + x \wedge y.$$

The symmetric form

$$x \cdot y = \sum_{j=1}^{p} x_j y_j - \sum_{j=p+1}^{d} x_j y_j = \mathcal{B}(x, y),$$

can be associated to the Euclidean inner product in the vectorial space $\mathbb{R}^d$ when $p = 0$ or $q = 0$. Therefore, $x \cdot y$ is defined as the *Clifford inner product* between the vectors x and y. We say that two vectors x and y are orthogonal whenever $x \cdot y = 0$.

In a similar way, the anti-symmetric part of the product xy defines a bi-vectorial part given by

$$x \wedge y = \frac{xy + yx}{2} = \sum_{i<y} (x_i x_j - x_j x_i) e_i e_j.$$

Again, when $p = 0$ or $q = 0$ and $d = 3$ it can be related to the outer (or wedge) Gibbs product. Hence, $x \wedge y$ is called the *Clifford outer product*. By convention, we say that x and y are collinear if and only if $x \wedge y = 0$.

The definition of inner and outer products acting on vectors may be extended in a straightforward way to the general case of r- and s-vectors as

$$x^r \cdot y^s = \begin{cases} [x^r y^s]_{|r-s|}, & \text{if } r, s > 0 \\ 0, & \text{if } r = 0 \text{ ou } s = 0 \end{cases} \quad \text{and} \quad x^r \wedge y^s = \begin{cases} [x^r y^s]_{r+s}, & \text{if } r + s \leq d; \\ 0, & \text{if } r + s > d. \end{cases},$$

respectively, where $x^r \in \mathbb{R}^r_{p,q}$ and $y^s \in \mathbb{R}^s_{p,q}$.

As one of the main interest of Clifford algebras is the description of orthogonal transformations, we now present the link with the orthogonal group $O(p, q)$ acting on $\mathbb{R}^{p,q}$. We recall that $O(p, q)$ denotes the group of orthogonal transformations that is, of all transformations $T : \mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$, for which

$$T(x) \cdot T(y) = x \cdot y, \quad \forall x, y \in \mathbb{R}^{p,q}.$$

Let A_T denote the matrix representation of the orthogonal transformation T with respect to the chosen basis $\{e_1, \cdots, e_d\}$. Then, $O(p, q)$ can be identified with the set of all $d \times d$ real invertible matrizes Q which satisfies to

$$Q^t A Q = A$$

and $\det Q = \pm 1$.

The elements of $O(p,q)$ such that its determinant is $+1$ are called rotations in $\mathbb{R}^{p,q}$ and if its determinat is -1 are called anti-rotations.

In order to obtain a group representation of $O(p,q)$ in $\mathbb{R}_{p,q}$ it is necessary to introduce a linear anti-automorphism called *conjugation*, defined by its action on the basic elements and extended by linearity the the whole of the Clifford algebra.

The conjugation $\bar{\cdot} : \mathbb{R}_{p,q} \to \mathbb{R}_{p,q}$, is then defined as

$$\overline{1} = 1, \quad \overline{e_i} = -e_i, i = 1, \cdots, d$$

and satisfying to

$$\overline{ab} = \bar{b}\bar{a}, \quad \overline{\lambda a + b} = \lambda \bar{a} + \bar{b},$$

for all $\lambda \in \mathbb{R}$ and $a, b \in \mathbb{R}_{p,q}$.

Given a vector $s \in \mathbb{R}^{p,q}$ we get $s\bar{s} = \bar{s}s = -\mathcal{B}(s,s)$. In particular we have

$$s\bar{s} = \bar{s}s = -\sum_{j=1}^{p} s_j^2 + \sum_{j=p+1}^{d} s_j^2 := \|s\|^2.$$

We remark, at this point, that although $\|\cdot\|$ does not represent a norm in general, it coincides with the Euclidean norm of the vector s in the case of $s \in \mathbb{R}^{0,d}$. Moreover, an element $s \in \mathbb{R}^{p,q}$ such that $s^2 = \mathcal{B}(s,s) \neq 0$ is invertible and its inverse is given by $s^{-1} = s/\|s\|^2$.

Let now $s \in \mathbb{R}^{p,q}$ with $\|s\| = 1$. We consider the associated linear transformation given by $x \mapsto \chi(s)x = sxs^{-1}$. First, we observe that $\chi(s) : \mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$. Indeed, for every $x \in \mathbb{R}^{p,q}$

$$\begin{aligned} \chi(s)x &= sxs^{-1} = sx\bar{s} \\ &= s(x^{\|} + x^{\perp})\bar{s} = (x^{\|} - x^{\perp})s\bar{s} \\ &= x^{\|} - x^{\perp} \in \mathbb{R}^{p,q}. \end{aligned}$$

Then $\chi(s)$ is an orthogonal transformation, that is $\chi(s) \in O(p,q)$, since

$$\begin{aligned} \|\chi(s)x\|^2 &= -\mathcal{B}(\chi(s)x, \chi(s)x) = \overline{sxs^{-1}}sxs^{-1} \\ &= \overline{sx\overline{s}}sx\overline{s} = (s\overline{xs})sx\overline{s} \\ &= s\overline{x}x\overline{s} = \|x\|^2 s\overline{s} = \|x\|^2. \end{aligned}$$

as, we recall, $\|s\| = 1$.

Hence, the action of $\chi(s) \in O(p,q)$ corresponds to a reflection with respect to the hyperplane $H_s = \{x \in \mathbb{R}^{p,q} : s \cdot x = 0\}$. Thus, by means of the Cartan-Dieudonné Theorem, any $T \in O(p,q)$ can be seen as the composition of a finite number of reflections with respect to hyperplanes H_{s_i}, where $s_i \in \mathbb{R}^{p,q}$ are invertible unitary vectors, that is $T \in O(p,q)$ can be represented by

$$\chi(\prod_{i=1}^{k} s_i) := \chi(s_k) \circ \cdots \circ \chi(s_1) : \mathbb{R}^{p,q} \to \mathbb{R}^{p,q},$$

and, therefore, every $T \in O(p,q)$ is isomorphic to $\prod_{i=1}^{k} s_i$, that is

$$\mathrm{Pin}(p,q) = \left\{ \prod_{i=1}^{k} s_i : s_i \in \mathbb{R}^{p,q} \text{ and } \|s_i\| = 1, i = 1, \cdots, k,\ k \in \mathbb{N} \right\}$$

defines a (two-fold) representation of the orthogonal group into the Clifford algebra $\mathbb{R}_{p,q}$, the so-called vectorial representation.

1.1.4 Complex Clifford Algebras

We define the complexified Clifford algebra $\mathbb{C}_d$ as the tensor product

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}_{p,q},$$

where $p + q = d$, that is, the algebra generated by the orthonormal basis

$$\{1 \otimes e_j,\ j = 1, \ldots, d\},$$

and associated to the complexified non-degenerate symmetric bilinear form $\mathcal{B}_{\mathbb{C}}(z,w) = \sum_{j=1}^{d} z_j w_j$. From now on, we denote the basis elements as $e_j = 1 \otimes e_j$. One immediate consequence of such a construction is the fact that $\mathbb{C}_d$ has no signature. Indeed, if the element e_j as signature -1, that is, $e_j^2 = -1$, then the linearly dependent element $e'_j := ie_j$ has symmetric signature, that is $e'^2_j = +1$.

The complexified Clifford algebra $\mathbb{C}_d$ is a linear associative algebra over $\mathbb{C}$ and it has complex dimension 2^d, and any Clifford number $w \in \mathbb{C}_d$ may thus be written as

$$w = \sum_A w_A e_A, \quad w_A \in \mathbb{C}.$$

The conjugation introduced on $\mathbb{R}_{p,q}$ can now be extended to $\mathbb{C}_d$ while keeping the same notation. The conjugation in $\mathbb{C}$ becomes

$$w \mapsto \overline{w} = \sum_A \overline{w}_A \overline{e}_A,$$

where $\overline{w}_A$ denotes the usual complex conjugation and $\overline{e}_A$ denotes the usual conjugation in $\mathbb{R}_{p,q}$. More important, for a vector $w = \sum_{j=1}^d w_j e_j$ we have

$$w\overline{w} = \|w\|^2 := \sum_{j=1}^d |w_j|^2,$$

the Euclidean norm of $w = (w_1, \ldots, w_d) \in \mathbb{C}^d$. Hence, each non-zero vector $w = \sum_{j=1}^d w_j e_j$ has an unique multiplicative inverse given by

$$w^{-1} = \frac{\overline{w}}{\|w\|^2}.$$

We finalize this subsection with the description of $\mathbb{C}^d$-valued functions. Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain. We now consider $\mathbb{C}^d$-valued functions $f : \Omega \subset \mathbb{R}^d \to \mathbb{C}^d$ which are represented as

$$z \mapsto f(z) = \sum_A f_A(z) e_A,$$

with $f_A : \Omega \to \mathbb{C}$. Properties such as continuity, L^2, etc. will be understood component-wisely. For example, $f \in L^2(\Omega)$ if and only if all components f_A belongs to $L^2(\Omega)$.

1.2 Monogenic Functions

Monogenic functions arise as generalisations of holomorphic functions in complex analysis to higher dimension. They are defined as the null solutions of the massless Dirac operator in d dimensions.

In this section we begin by presenting the definition of monogenicity followed by some of its more general properties. Then we introduce the Fischer inner product and prove the Fischer decomposition for spaces of polynomials of fixed degree in terms of inner monogenics. We end this section with the Cauchy-Kovalevska extension.

1.2.1 Definitions and Properties

Definition 1.2. *Let $\underline{\Omega}$ be an open domain in $\mathbb{R}^d$. A C^1-function $f : \underline{\Omega} \subset \mathbb{R}^d \to \mathbb{R}_{p,q}$ is said to be left monogenic (respect., right monogenic) if it is a null solution of the (massless) Dirac equation*

$$\partial_{\underline{x}} f = 0 \ (respect., \ f\partial_{\underline{x}} = 0), \quad in \ \underline{\Omega},$$

where $\partial_{\underline{x}} = \sum_{j=1}^{d} \partial_{x_j} e_j$.

The massless Dirac operator factorizes the $d-$Laplacian, that is

$$\partial_{\underline{x}}^2 f = -\Delta_d f.$$

Alternatively, we define left monogenic (respect. right monogenic) functions as C^1 null-solutions of the Cauchy-Riemann operator operator

$$Df = (\partial_{x_0} + \partial_{\underline{x}})f = 0, \quad (respect. \ fD = 0),$$

in $\Omega \in \mathbb{R}^{d+1} = \mathbb{R} \oplus \mathbb{R}^d$. Similarly to the above, the Cauchy-Riemann operator factorizes the $(d+1)$-dimensional Laplace operator in the sense that

$$\overline{D}D = D\overline{D} = -\Delta_{d+1},$$

with $\overline{D} := \partial_{x_0} - \partial_{\underline{x}}$.

Taking polar coordinates $\underline{x} = r\xi$, with $r = |\underline{x}|$ and $\xi \in S^{d-1}$, one may write the Dirac operator $\partial_{\underline{x}}$ as

$$\begin{aligned} \partial_{\underline{x}} &= \xi\partial_r + \frac{1}{r}\partial_\xi \\ &= \frac{\xi}{r}\left(r\partial_r + \overline{\xi}\partial_\xi\right), \end{aligned} \tag{1.1}$$

where ∂_ξ is a differential operator depending only on the angular coordinates. Hence, we define the spherical Dirac operator as

$$\Gamma_\xi = \overline{\xi}\partial_\xi. \tag{1.2}$$

These operators satisfy the Weyl relations

$$[\partial_r, \xi] = \partial_r\xi - \xi\partial_r = 0$$

and

$$[\partial_r, \Gamma_\xi] = \partial_r\Gamma_\xi - \Gamma_\xi\partial_r = 0.$$

Since $\underline{\overline{x}}\partial_{\underline{x}} = r\partial_r + \Gamma_\xi$ we obtain that the spherical Dirac operator can be written as

$$\begin{aligned}\Gamma_\xi &= \overline{x} \wedge \partial \\ &= -\sum_{i<j} e_i e_j \left(x_i \partial_{x_j} - x_j \partial_{x_i}\right).\end{aligned}$$

We now draw our attention to the case of homogenous monogenic functions and its restrictions to the unit sphere S^{d-1}- the so-called spherical monogenic functions. First, we consider the space $\mathcal{H}(k;\mathbb{C}_d)$ of $\mathbb{C}_d-$valued harmonic polynomials, homogeneous of degree $k \in \mathbb{N}$, and restricted to the unit sphere S^{d-1}.

Definition 1.3. *Given $k \in \mathbb{N}$, we denote by $M^+(k,\mathbb{C}_d)$ the space of homogeneous monogenic $\mathbb{C}_d$-valued polynomials of degree k in $\mathbb{R}^d$. The elements of $M^+(k,\mathbb{C}_d)$ will be called inner monogenics of order k.*

The inversion operator $f \mapsto \mathbb{I}(f)$, where

$$x \mapsto \mathbb{I}(f)(x) = \frac{\overline{x}}{\|x\|^d} f\left(\frac{\overline{x}}{\|x\|^2}\right), \quad x \neq 0$$

maps an inner monogenic of order k into a monogenic function homogeneous of degree $-(k+d)$.

Definition 1.4. *Given $k \in \mathbb{N}$, we denote by $M^-(k,\mathbb{C}_d)$ the space of homogeneous monogenic $\mathbb{C}_d$-valued functions of degree $-(k+d)$ in $\mathbb{R}^d \setminus \{0\}$. The elements of $M^-(k,\mathbb{C}_d)$ are called outer monogenics of order k.*

We denote by $\mathcal{M}^\pm(k,\mathbb{C}_d)$ the space of the restrictions to the unit sphere S^{d-1} of the elements of $M^\pm(k,\mathbb{C}_d)$, respectively. Their elements are called *inner and outer spherical monogenics*, respectively. We remark that these elements are polynomials.

Theorem 1.1 (see c.f. [31], pag. 157)**.**

1. *For each $k \in \mathbb{N}$, $P_k \in \mathcal{M}^+(k,\mathbb{C}_d)$ is a $\mathbb{C}_d$-valued spherical harmonic of order k, that is*

$$\Gamma_\xi P_k(\xi) = -kP_k(\xi), \quad \Delta_\xi P_k(\xi) = -k(k+d-2)P_k(\xi);$$

2. *For each $l \in \mathbb{N}$, $Q_l \in \mathcal{M}^-(k,\mathbb{C}_d)$ is a $\mathbb{C}_d$-valued spherical harmonic of order $l+1$, that is*

$$\Gamma_\xi Q_l(\xi) = (l+d-1)Q_l(\xi), \quad \Delta_\xi Q_l(\xi) = -(l+1)(l+d-1)Q_l(\xi);$$

3. *For all $k \in \mathbb{N}$*

$$\mathcal{H}(k,\mathbb{C}_d) = \mathcal{M}^+(k,\mathbb{C}_d) \oplus \mathcal{M}^-(k,\mathbb{C}_d).$$

1.2.2 Fischer Decomposition

The Fischer decomposition can be traced back to Ernst Sigismund Fischer [10] in 1917. Basically, it shows that under certain conditions a polynomial can be decomposed in terms of homogeneous polynomial solutions of a given elliptic operator. Originally proved for the case of harmonic polynomials it was later on extended to the case of the Dirac operator [31].

Definition 1.5 (Fischer inner product). *Let $\mathcal{P}(k)$ denote the space of homogeneous polynomials of degree $k \in \mathbb{N}_0$. We define the Fischer inner product (with respect to the Laplace operator) between two elements $P, Q \in \mathcal{P}(k)$ as*

$$\langle P, Q \rangle_k := \left[\overline{P(\partial)} Q(x) \right]_0 \Big|_{x=0}, \tag{1.3}$$

where $P(\partial)$ is the differential operator obtained from P by replacing the variable x_i by ∂_{x_i}.

Using this inner product with respect to Dirac operator ∂ we can decompose the space of homogeneous polynomials in terms of inner spherical monogenics.

Lemma 1.2. *Let $k \in \mathbb{N}$. Then, for all $P \in \mathcal{P}(k-1)$ and $Q \in \mathcal{P}(k)$ it holds*

$$\langle xP, Q \rangle_k = -\langle P, \partial Q \rangle_{k-1},$$

that is to say, the space M_k^+ of monogenic homogeneous polynomials of degree k is orthogonal to $x\mathcal{P}(k-1)$.

This leads to the Fischer decomposition theorem (for monogenic polynomials).

Theorem 1.3. *The space of homogeneous polynomials of degree $k \in \mathbb{N}_0$ admits the orthogonal decomposition*

$$\mathcal{P}(k) = \sum_{n=0}^{k} \oplus x^n M_{k-n}^+. \tag{1.4}$$

For the proof we refer to [31].

1.2.3 Cauchy-Kowalevskaya Extension

The Cauchy-Kovalevskaya extension theorem ensures local existence and uniqueness for solutions of partial differential equations with analytic initial value data. It has been elegantly generalised to higher dimension in the framework of Clifford analysis. In this overview we restrict ourselves to the case of the Dirac operator, that is, when the theorem states that a monogenic function in an appropriated domain is completely determined by its restriction to a $(d-1)-$hyperplane.

We begin by the definition of a *normal neighbourhood.* Let $\underline{\Omega} \subset \mathbb{R}^d$ be an open set. An open neighbourhood Ω of $\underline{\Omega}$ in $\mathbb{R}^{d+1}$ is said to be x_0-normal if for each $x \in \Omega$ the line segment $\{x + te_0 : t \in \mathbb{R}\} \cap \Omega$ is connected and contains just one point in $\underline{\Omega}$.

Theorem 1.4. *Let $\underline{\Omega}$ be an open and connected set in $\mathbb{R}^d$. Given $f \in \mathcal{A}(\underline{\Omega}, \mathbb{C}_d)$, the set of $\mathbb{C}_d-$valued analytic functions in $\underline{\Omega}$, the function F given by*

$$F(x) = \sum_{k=0}^{\infty} \frac{1}{k!}(-x_0)^k (\bar{e}_0 \partial_x)^k f(\underline{x}) = e^{-x_0 \partial_x} f(x). \tag{1.5}$$

satisfies $\partial_x F = 0$ in an open connected and normal neighbourhood Ω of $\underline{\Omega}$ in $\mathbb{R}^{d+1}$. Moreover $F\big|_{x_0=0} = f$ in Ω.

The proof for this theorem can be found in [31]. We now consider the family of all such monogenic extensions of f that admits a maximal element. Indeed, take the family of all couples $(\Omega_\alpha, F_\alpha)$ where Ω_α is an open connected and normal neighbourhood of $\underline{\Omega}, F_\alpha$ is monogenic at left and $F_\alpha|_{\Omega} f$. Then this family is non-trivial and it satisfies

$$(\Omega_\alpha, F_\alpha) \leq (\Omega_\beta, F_\beta), \quad \text{for } \alpha \leq \beta$$

and

$$\Omega_\alpha \subset \Omega_\beta \Rightarrow F_\beta|_{\Omega_\alpha} = F_\alpha.$$

Therefore, it has a maximal element (Ω, F) (see [31]) with

$$\Omega = \cup_{\alpha \in A} \Omega_\alpha$$

and

$$F|_{\Omega_\alpha} = F_\alpha.$$

Moreover, (Ω, F) satisfies Theorem 1.4 and it is called the *Cauchy-Kowalevskaya extension* of $(\underline{\Omega}, f)$. For convenience we say F is the *CK-extension* of f.

This theorem extends in an easy way to the case of the Cauchy-Riemann operator. Indeed, for $f \in \mathcal{A}(\underline{\Omega}, \mathbb{C}_d)$ its CK-extension w.r.t. the Cauchy-Riemann operator is given by

$$\begin{aligned} F(x) &= e^{-x_0 \partial_{\underline{x}}} f(\underline{x}) \\ &= \sum_{k=0}^{\infty} \frac{(-1)}{k!} (x_0)^k (\partial_x)^k f(\underline{x}). \end{aligned}$$

1.3 Reproducing Kernel Hilbert Modules

In this subsection we introduce a useful family of inner product spaces called reproducing kernel Hilbert spaces (RKHS). Each of these spaces is associated to a particular kernel, and its applications range from determination of approximate solutions for linear singular integral equations to the theory of Stochastic Processes.

1.3.1 Clifford Hilbert Modules

Due to the anti-commutativity of the Clifford algebras we cannot consider usual Hilbert spaces of functions. For example, left monogenicity is preserved under multiplication at right by a Clifford constant but, in general, not by multiplication at left. This motivates the structure of *Clifford Hilbert module.*

Definition 1.6. *A vector space V is said a left (unitary) module over $\mathbb{C}_d$ (or left $\mathbb{C}_d$-module) if there exists an algebra morphism $L : \mathbb{C}_d \to End(V)$, that is, for all $a \in \mathbb{C}_d$ we have a linear transformation $L(a)$ such that*

$$L(ab + c) = L(a)L(b) + L(c),$$

and where $L(1)$ is the identity operator. Moreover, $L(a)$ is called left multiplication.

In the same way, we can define a right (unitary) module over $\mathbb{C}_d$.

Definition 1.7. *A right (unitary) module over $\mathbb{C}_d$ (or right $\mathbb{C}_d$-module) is a vector space which there exists an algebra morphism $R : \mathbb{C}_d \to End(V)$, that is, for all $a \in \mathbb{C}_d$ we have a linear transformation $R(a)$ such that*

$$R(ab + c) = R(b)R(a) + R(c),$$

and where $R(1)$ is the identity operator. Moreover, $R(a)$ is called right multiplication.

A bi-module is defined as a module which is both a left- and a right-module and then, we have that a left and right multiplication commute, i.e.

$$L(a)R(b) = R(b)L(a).$$

As an example, let us consider a $\mathbb{C}_d$-vector space V. We define the corresponding left $\mathbb{C}_d-$module V_d as

$$V_d := \mathbb{C} \otimes V = \{x = \sum_A x_A \otimes e_A;\ x_A \in \mathbb{C}\}$$

where $A = \{l_1, l_2, \ldots, l_r\} \subseteq M = \{1, \ldots, d\}$ with $1 \leq l_1 < \ldots < l_r \leq d$ and $\mathbb{C}_d$ acts on V_d as

$$ax \quad = \quad \sum_{A,B} a_A x_B \otimes e_A e_B,$$

where $x \in V_d$ and $a \in \mathbb{C}_d$. Similarly, we define the right $\mathbb{C}_d$-Clifford module where $\mathbb{C}_d$ acts on V_d by

$$xa \;=\; \sum_{A,B} a_A x_B \otimes e_B e_A, \quad x \in V_d, a \in \mathbb{C}_d.$$

Now, given a complex Hilbert space $\mathcal{H}$ with a inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, we define the right unitary module $\mathcal{H}_d$ as

$$\mathcal{H}_d := \mathcal{H} \otimes \mathbb{C}_d$$

where its elements have the form $x = \sum_A x_A e_A$ with $x_A \in \mathcal{H}$ for all $A = \{l_1, l_2, \ldots, l_r\} \subseteq M = \{1, \ldots, d\}$ with $1 \leq l_1 < \ldots < l_r \leq d$. Moreover, $\mathcal{H}_d$ is a complex Hilbert space equipped with the scalar product

$$\begin{aligned} (\cdot, \cdot) \;&:\; \mathcal{H}_d \times \mathcal{H}_d \to \mathbb{C} \\ (x, y) \;&:=\; \sum_A \langle x_A, y_A \rangle_{\mathcal{H}}, \end{aligned} \tag{1.6}$$

$x, y \in \mathcal{H}_d$.

Definition (1.6) induces the following Euclidean metric on $\mathcal{H}_d$:

$$||x||^2_{\mathcal{H}_d} := (x, x) = \sum_A ||x_A||^2. \tag{1.7}$$

Since the definition of scalar product is not Clifford algebra-valued, we now introduce a complexified Clifford algebra-valued product on $\mathcal{H}_d$ as

$$\begin{aligned} \langle \cdot, \cdot \rangle \;&:\; \mathcal{H}_d \times \mathcal{H}_d \to \mathbb{C}_d \\ \langle x, y \rangle \;&:=\; \sum_{A,B} \langle x_A, y_A \rangle_{\mathcal{H}} e_A \overline{e}_B, \end{aligned} \tag{1.8}$$

where the definition of conjugation on $\mathcal{H}_d$ is given from the conjugation on $\mathcal{H}$ as

$$\begin{aligned} \bar{\cdot} \;&:\; \mathcal{H}_d \to \mathcal{H}_d \\ x \;&\mapsto\; \sum_A \overline{x_A}\; \overline{e_A}. \end{aligned}$$

While the inner product (1.6) gives rise to a norm, definition (1.8) provides a generalization of Riesz' representation theorem in the sense that a linear functional ψ is continuous if and only if it can be represented by an element $f_\psi \in \mathcal{H}_d$ such that

$$\psi(g) = \langle f_\psi, g \rangle.$$

The right $\mathbb{C}_d$-module $\mathcal{H}$ is called the right Hilbert $\mathbb{C}_d$-module if it is provided with an inner product such that it is complete with the norm topology derived from this inner product.

1.3.2 Reproducing Kernel Hilbert Modules

Definition 1.8. *Let $\mathcal{H}$ be a right Hilbert $\mathbb{C}_d$-module with an inner product $\langle\, ,\rangle$ consisting of $\mathbb{C}_d$-valued functions f defined on some set E. A function $K : E \times E \to \mathbb{C}_d$ is called a reproducing kernel of $\mathcal{H}$ if for any $t \in E$ fixed,*

(i) $K(\cdot, t) \in \mathcal{H}$;

(ii) $f(t) = \langle K(\cdot, t), f(\cdot)\rangle$ *for all* $f \in \mathcal{H}$.

Moreover, $\mathcal{H}$ is then said to be a right Hilbert $\mathbb{C}_d$-module with reproducing kernel K, or reproducing kernel Hilbert module.

One may observe that, when $d = 1$, any reproducing kernel Hilbert module is a Hilbert space with a reproducing kernel.

Theorem 1.5. *$\mathcal{H}$ is a right Hilbert $\mathbb{C}_d$-module (of $\mathbb{C}_d$-valued functions defined on E) with reproducing kernel K if and only if for any $t \in E$, there exists a constant $C(t) > 0$ such that*

$$|f(t)| \le C(t)||f||$$

for all $f \in \mathcal{H}$.

Proof. Indeed, if such kernel K exists, then for every $t \in E$, we have

$$f(t) = \langle K(\cdot, t), f(\cdot)\rangle,$$

so that

$$|f(t)| = |\langle K(\cdot, t), f(\cdot)\rangle| \le 2^d ||K(\cdot, t)||\ ||f||.$$

We now consider $C(t)$ as a positive constant depending on $t \in E$ such that

$$|f(t)| \le C(t)||f||$$

for all $f \in \mathcal{H}$, and take the right $\mathbb{C}_d$-linear fuctional T_f on $\mathcal{H}$ for which

$$T_f(f) = f(t)$$

for all $f \in \mathcal{H}$. Thus, T_f is bounded and then, from the Riesz representation theorem, there exists a unique element $h_f \in \mathcal{H}$ with

$$f(t) = T_f(f) = \langle h_f, f\rangle$$

for all $f \in \mathcal{H}$. Taking $K(\cdot, t) := h_t$, we obtain the desired result, namely, that K is a reproducing kernel of $\mathcal{H}$.

$\square$

Chapter 2

Fractional Calculus

Fractional calculus has wide applications in diverse fields of engineering and science such as optics, signals processing, electromagnetics, viscoelasticity, fluid mechanics, to name just a few ([32]). Indeed, the fractional concept of derivative has been used to model physical processes (in classical mechanics, hadron spectroscopy, etc.) and in consequence, new interesting and useful results were found.

There are several possible definitions of *fractional derivatives* $\frac{d^\alpha}{dx^\alpha}$, where α denotes a (not any more a necessarily integer) real number. Obviously, the choice for α depends on the intended application. For instance, the $3-$body problem requires $\alpha = 1/3$. However, we do expect all possible definitions to share some common aspects; for example, they must coincide with the classic derivatives whenever α is a positive integer.

On the other hand, it is also expected that well-established techniques used in differential calculus may not be easily transferred (and in some cases, cannot be transferred) to the field of fractional calculus. A good example is the Leibniz rule which cannot be fulfilled by true fractional derivatives (see [37]). Indeed, the fractional derivative of a product of functions satisfies the generalized Leibniz rule given by

$$\frac{d^\alpha}{dx^\alpha}(fg) = \sum_{j=0}^{\infty} \binom{\alpha}{j} \frac{d^{\alpha-j}}{dx^{\alpha-j}}(f)\frac{d^j}{dx^j}(g).$$

Several definitions of fractional derivatives exist depending on the objectives and intended applications. The more common definitions are the *Riemann-Liouville derivative* and the *Caputo derivative*, but there exist many more. Also, different definitions lead to different properties and results. For example, the Caputo derivative of a constant is still zero, but that is no longer the case for the Riemann-Liouville derivative. Of course, the choice of the fractional derivative to use in each problem is also influenced by the problem itself. We will discuss this point further on when presenting the definitions of these derivatives. Before introducing the above mentioned fractional derivatives, namely, the Riemann-Liouville derivative and the Caputo derivative, we make a small recapitulation of basic, yet necessary, facts.

For more details about this theory, we refer the interested reader to [22, 32, 35, 38].

2.1 Gamma and the Generalized Hypergeometric Functions

Essential for the good understanding of this chapter is the Gamma function as defined by Euler, as well as some of its properties and the generalized hypergeometric functions.

Definition 2.1 (Gamma function). *The Gamma function is defined as* $\Gamma : \mathbb{C} \setminus \{z : \Re(z) > 0\} \to \mathbb{C}$ *by the so-called Euler integral of the second kind*

$$z \mapsto \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt. \tag{2.1}$$

This leads to some useful properties (see [35])

1. For $\Re(z) > 0$ and $\Re(w) > 0$, we have

$$\Gamma(z)\Gamma(w) = \Gamma(z+w)B(z,w),$$

   where $B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt$ denotes the *Beta function.*

2. $\Gamma(1) = 1$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$;

3. $\Gamma(z)\Gamma(1-z) = B(z, 1-z) = \frac{\pi}{\sin(\pi z)}$, for $0 < \Re(z) < 1$;

4. $\Gamma(z+1) = z\Gamma(z)$ again for $\Re(z) > 0$;

More important, the Gamma function can be analytically continued to a meromorphic function in the plane, with a continuous inverse and having simple poles in $\mathbb{Z}_0^- = \{0, -1, -2, \cdots\}$. Moreover, it acts as an extension of the factorial function, since $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}_0$.

Definition 2.2 (Pochhammer symbol). *The Pochhammer symbol, or rising factorial, is defined for all z as*

$$(z)_0 = 1, \quad (z)_n := z(z+1)(z+2)\cdots(z+n-1), \quad n = 1, 2, 3, \cdots \tag{2.2}$$

Again, we list some of its properties (see [35]). For all $n, m \in \mathbb{N}$ we have

1. $(z)_{n+m} = (z)_m + (z+m)_n$;

2. $\binom{n}{k} = (-1)^k \frac{(-n)_k}{k!}$ for $k = 0, 1, \cdots, n$;

3. $\dfrac{\Gamma(z+n)}{\Gamma(z)} = (z)_n$, for $\Re(z) > 0$.

We finalize this subsection with the generalized hypergeometric functions and a special case of the *Meijer G-function.* This particular function is defined by means of a Mellin-Barnes type contour integral.

Definition 2.3. *1. The generalized hypergeometric function ${}_pF_q$ is defined via the power series*

$$ {}_pF_q\left(\begin{matrix} a_1 \cdots a_p \\ b_1 \cdots b_q \end{matrix} \Big| z \right) = \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{p}(a_k)_n}{\prod_{k=1}^{q}(b_k)_n} \frac{z^n}{n!}, \tag{2.3} $$

over the domain of convergence

$$ |z| < \infty \quad \textit{for } p \leq q $$

and

$$ |z| < 1 \quad \textit{for } p = q+1, $$

where $a_j, b_l \in \mathbb{C}, b_l \neq 0, -1, -2, \cdots$ $(j = 1, \cdots, p; l = 1, \cdots, q)$ and $(a)_n$ is the Pochhammer symbol (2.2).

2. Let $G^{m,n}_{p,q}$ be the Meijer G-function which is defined by means of the contour Mellin-Barnes type integral

$$ \begin{aligned} G^{m,n}_{p,q}\left[z\Big| \begin{matrix} (a_k)_1^p \\ (b_k)_1^q \end{matrix} \right] &= G^{m,n}_{p,q}\left[z\Big| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \right] \\ &= \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\prod_{k=1}^{m}\Gamma(b_k - s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{k=m+1}^{q}\Gamma(1-b_k+s)\prod_{j=n+1}^{p}\Gamma(a_j - s)} z^s ds \end{aligned} $$

where $z \neq 0$ is a complex variable, the integers $0 \leq m \leq q$, $0 \leq n \leq p$ define the order $(m, n; p, q)$ of the G-function. the parameters $(a_j)_1^p$, $(b_k)_1^q$ are such that none of the poles of $\Gamma(b_k - s)$, $k = 1, \ldots, m$ coincide with any of the poles of $\Gamma(1 - a_j + s)$, $k = 1, \ldots, n$ and the infinite contour $\mathcal{L}$ is situated in the complex plane in such a manner that it separates the poles of these two sets of Γ-functions in the numerator of the integrand.

2.2 Riemann-Liouville and Caputo Fractional Derivatives

In the next two subsections we present two of the most common fractional derivatives namely, the *Riemann-Liouville derivative* and the *Caputo derivative.* These two derivatives are constructed by a combination of the classical derivative with a singular integral operator.

2.2.1 Riemann-Liouville Derivatives

Definition 2.4. *Let $\Omega = [a,b]$, $-\infty < a < b < \infty$, be a finite interval on the real axis $\mathbb{R}$ and consider the space of summable and continuous functions in Ω. Given f in that space its left-sided Riemann-Liouville fractional integral $I^{\alpha}_{a^+}f$ of order $\alpha \in \mathbb{C}$, with $\Re(\alpha) > 0$, is defined by*

$$I^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(w)dw}{(x-w)^{1-\alpha}} \quad (x > a), \tag{2.4}$$

where $\Gamma(\alpha)$ is the defined by 2.1. In the same way, we define the right-sided Riemann-Liouville fractional integral $I^{\alpha}_{b^-}f$ of order $\alpha \in \mathbb{C}$, with $\Re(\alpha) > 0$, as

$$I^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(w)dw}{(w-x)^{1-\alpha}} \quad (x < b), \tag{2.5}$$

We remark that these definitions are dual and the results for left- and right-side operators are similar. Moreover, for $\alpha = n \in \mathbb{N}$ we have that (2.4) and (2.5) coincide with

$$I^{n}_{a^+}f(x) = \frac{1}{(n-1)!}\int_a^x \frac{f(w)dw}{(x-w)^{n-1}}, \quad I^{n}_{b^-}f(x) = \frac{1}{(n-1)!}\int_x^b \frac{f(w)dw}{(w-x)^{n-1}},$$

which can be understood as iterated integrals.

In what follows we restrict ourselves to the case of left-sided operators (2.4).

We now define the Riemann-Liouville fractional derivative operator.

Definition 2.5. *The Riemann-Liouville fractional derivative $\mathcal{D}^{\alpha}_{a^+}f$ of order $\alpha \in \mathbb{C}$ ($\Re(\alpha) \geq 0$) is defined by*

$$\mathcal{D}^{\alpha}_{a^+}f(x) := \left(\frac{d}{dx}\right)^n (I^{n-\alpha}_{a^+}f)(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x \frac{f(t)dt}{(x-t)^{\alpha-n+1}} \quad (x > a), \tag{2.6}$$

where n is given by

$$n = n(\alpha) := \begin{cases} \lfloor \Re(\alpha) \rfloor + 1, & \text{for } \alpha \notin \mathbb{N}_0; \\ \alpha, & \text{for } \alpha \in \mathbb{N}_0. \end{cases} \tag{2.7}$$

In particular, for $0 < \alpha < 1$, the Riemann-Liouville fractional derivative operator of order α becomes

$$\mathcal{D}^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(t)dt}{(x-t)^{\alpha}} \quad (x > a)$$

It may be directly verified the following properties of Riemann-Liouville fractional operators of order α.

Theorem 2.1. *If $\beta \in \mathbb{C}$ is such that $\Re(\beta) > 0$, then*

$$I^{\alpha}_{a^+}(x-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1} \tag{2.8}$$

while

$$\mathcal{D}^{\alpha}_{a^+}(x-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1} \tag{2.9}$$

The Riemann-Liouville fractional integrals satisfy the semigroup property.

Lemma 2.2. *Equations*

$$I^{\alpha}_{a^+}I^{\beta}_{a^+}f(x) = I^{\alpha+\beta}_{a^+}f(x) \quad \text{and} \quad I^{\alpha}_{b^-}I^{\beta}_{b^-}f(x) = I^{\alpha+\beta}_{b^-}f(x) \tag{2.10}$$

are satisfied at almost every point $x \in [a,b]$ *for* $f \in L_p(a,b)$ $(1 \le p \le \infty)$. *If* $\alpha+\beta > 1$, *then relations (2.10) hold at every point of* $[a,b]$.

For the proof we refer to [35]. However, the above result cannot be extended in general to the Riemann-Liouville fractional derivative operators. In the next theorem we present conditions under which this result does hold.

Theorem 2.3. *Let* $\alpha, \beta > 0$ *and* $m, n \in \mathbb{N}$ *such that* $m-1 \le \alpha < m$ *and* $n-1 \le \beta < n$. *Given* $\mathcal{D}^{\alpha}_{a^+}$ *and* $\mathcal{D}^{\beta}_{a^+}$ *are fractional Riemann-Liouville derivative operators the composition* $\mathcal{D}^{\alpha}_{a^+}\mathcal{D}^{\beta}_{a^+}f(t)$ *is given by*

$$\mathcal{D}^{\alpha}_{a^+}\mathcal{D}^{\beta}_{a^+}f(t) = \mathcal{D}^{\alpha+\beta}_{a^+}f(t) - \sum_{j=1}^{n}[\mathcal{D}^{\beta-j}_{a^+}f(t)]_{t=a}\frac{(t-a)^{-\alpha-j}}{\Gamma(-\alpha-j+1)} \tag{2.11}$$

Proof. From the three following equalities (see [35]):

$$\begin{aligned}
\mathcal{D}^{\alpha}_{a^+}f(t) &= \frac{d^k}{dt^k}\mathcal{D}^{-(k-\alpha)}_{a^+}f(t), \quad (k-1 \le \alpha < k)\\
\mathcal{D}^{-\alpha}_{a^+}\mathcal{D}^{\beta}_{a^+}f(t) &= \mathcal{D}^{\beta-\alpha}_{a^+}f(t) - \sum_{j=1}^{k}[\mathcal{D}^{\beta-j}_{a^+}f(t)]_{t=a}\frac{(t-a)^{\alpha-j}}{\Gamma(\alpha-j+1)}\\
\text{and} \quad \frac{d^n}{dt^n}\mathcal{D}^{\alpha}_{a^+}f(t) &= \mathcal{D}^{n+\alpha}_{a^+}f(t)
\end{aligned}$$

we obtain

$$\begin{aligned}
\mathcal{D}^{\alpha}_{a^+}\mathcal{D}^{\beta}_{a^+}f(t) &= \frac{d^m}{dt^m}\{\mathcal{D}^{-(m-\alpha)}_{a^+}\mathcal{D}^{\beta}_{a^+}f(t)\}\\
&= \frac{d^m}{dt^m}\{\mathcal{D}^{\alpha+\beta+m}_{a^+}f(t) - \sum_{j=1}^{k}[\mathcal{D}^{\beta-j}_{a^+}f(t)]_{t=a}\frac{(t-a)^{m-\alpha-j}}{\Gamma(m-\alpha-j+1)}\}\\
&= \mathcal{D}^{\alpha+\beta}_{a^+}f(t) - \sum_{j=1}^{n}[\mathcal{D}^{\beta-j}_{a^+}f(t)]_{t=a}\frac{(t-a)^{-\alpha-j}}{\Gamma(-\alpha-j+1)},
\end{aligned}$$

as desired. □

Now, interchanging the roles of α and β in 2.11, we get

$$\mathcal{D}^{\beta}_{a^+}\mathcal{D}^{\alpha}_{a^+}f(t) = \mathcal{D}^{\beta+\alpha}_{a^+}f(t) - \sum_{j=1}^{n}[\mathcal{D}^{\alpha-j}_{a^+}f(t)]_{t=a}\frac{(t-a)^{-\beta-j}}{\Gamma(-\beta-j+1)} \tag{2.12}$$

Hence, we have that the Riemann-Liouville fractional derivative operators, $\mathcal{D}^{\alpha}_{a^+}$ and $\mathcal{D}^{\beta}_{a^+}$, commutes and that $\mathcal{D}^{\alpha}_{a^+}\mathcal{D}^{\beta}_{a^+} = \mathcal{D}^{\alpha+\beta}_{a^+}$ in the following cases:

- $\alpha = \beta$; (Trivial case)
- for $\alpha \neq \beta$ under the assumption that

$$\sum_{j=1}^{m}[\mathcal{D}^{\alpha-j}_{a^+}f(t)]|_{t=a} = \sum_{j=1}^{n}[\mathcal{D}^{q-j}_{a^+}f(t)]|_{t=a} = 0 \tag{2.13}$$

Let us consider the case where both $\alpha, \beta \in [0, 1[$ (that is to say, $m = n = 1$). Then $\mathcal{D}^{\alpha}_{a^+}$ and $\mathcal{D}^{\beta}_{a^+}$ commutes and we have $\mathcal{D}^{\alpha}_{a^+}\mathcal{D}^{\beta}_{a^+} = \mathcal{D}^{\alpha+\beta}_{a^+}$ since

$$\mathcal{D}^{\alpha-1}_{a^+}f(t)|_{t=a} = \mathcal{D}^{\beta-1}_{a^+}f(t)|_{t=a} = 0.$$

On the other hand, it is proved (see [35]) that

$$\mathcal{D}^{\alpha-1}_{a^+}f|_{t=a} = 0 \Leftrightarrow f(a) = 0. \tag{2.14}$$

Hence, from (2.14) we obtain

$$\mathcal{D}^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}f(x) = \mathcal{D}^{2\alpha}_{a^+}f(x) \quad \Leftrightarrow \quad f(a) = 0, \tag{2.15}$$

that is to say, $\left(\mathcal{D}^{\alpha}_{a^+}\right)^2 f = \mathcal{D}^{2\alpha}_{a^+}f$ if and only if $f(a) = 0$.

We conclude with the remark that the Riemann-Liouville derivative of a constant is no longer zero. In fact, taking $\beta = 1$ in (2.9), we obtain

$$\mathcal{D}^{\alpha}_{a^+}1(x) = \frac{1}{\Gamma(1-\alpha)}(x-a)^{-\alpha}.$$

This implies that although this fractional derivative produces nice theoretical results it is difficult to handle numerically due to instability issues (small changes in data may produce larger changes in output).

2.2.2 Caputo Derivatives

We now present the Caputo fractional derivative operator.

Definition 2.6. *Let $\alpha \in \mathbb{C}$ be such that $\Re(\alpha) \geq 0$, and let n be given by (2.7). The left-sided Caputo fractional derivative ${}^{C}\mathcal{D}^{\alpha}_{a^+}$ is defined by*

$$ {}^{C}\mathcal{D}^{\alpha}_{a^+} f(x) := I^{n-\alpha}_{a^+} \left(\frac{d}{dx}\right)^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(w)dw}{(x-w)^{\alpha-n+1}} \quad (x > a). $$

where f is an absolutely continuous function and $I^{n-\alpha}_{a^+}$ is the left-sided Riemann-Liouville fractional operator of order $n-\alpha$ defined in (2.4).

In particular, if $0 < \alpha < 1$, the left-sided Caputo fractional derivative operator of order α of a function f is given by

$$ {}^{C}\mathcal{D}^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f'(w)dw}{(x-w)^{\alpha}} \quad (x > a). \tag{2.16} $$

In the following theorem we show the connection between the Riemann-Liouville and the Caputo derivatives of same order α, $0 < \alpha < 1$.

Theorem 2.4. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function. Then, the left-sided Caputo fractional derivative of order α, $0 < \alpha < 1$ satisfy*

$$ {}^{C}\mathcal{D}^{\alpha}_{a^+} f(x) = \mathcal{D}^{\alpha}_{a^+} f(x) - \frac{f(a)}{\Gamma(1-\alpha)(x-a)^{\alpha}}, \quad x \in [a, b] \tag{2.17} $$

where $\mathcal{D}^{\alpha}_{a^+}$ is left-sided Riemann-Liouville fractional derivative of order α, $0 < \alpha < 1$.

Proof. By Definition 2.5, the left-sided Riemann-Liouville fractional derivative operator for $0 < \alpha < 1$ is given as

$$ \mathcal{D}^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(w)dw}{(x-w)^{\alpha}} \quad (0 < \alpha < 1; x > a) \tag{2.18} $$

Taking into account that

$$ \frac{d}{dw}[f(w)(x-w)^{1-\alpha}] = f'(w)(x-w)^{1-\alpha} - (1-\alpha)\frac{f(w)}{(x-w)^{\alpha}} $$

we obtain

$$ \int_a^x \frac{f(w)dw}{(x-w)^{\alpha}} = \frac{1}{1-\alpha}\Big\{ \int_a^x \frac{d}{dw}[-f(w)(x-w)^{1-\alpha}]dw + \int_a^x f'(w)(x-w)^{1-\alpha}dw \Big\}. $$

Using the Fundamental Theorem of Calculus we get

$$ \int_a^x \frac{f(w)dw}{(x-w)^{\alpha}} = \frac{1}{1-\alpha}\Big[f(a)(x-a)^{1-\alpha}\Big] + \frac{1}{1-\alpha}\int_a^x f'(w)(x-w)^{1-\alpha}dw \tag{2.19} $$

and, in consequence of Leibniz integral rule for standard derivatives, it follows that

$$\begin{aligned}
\frac{d}{dx}\int_a^x \frac{f(w)dw}{(x-w)^\alpha} &= \frac{d}{dx}\Big\{\frac{1}{1-\alpha}\Big[f(a)(x-a)^{1-\alpha}\Big] + \frac{1}{1-\alpha}\int_a^x f'(w)(x-w)^{1-\alpha}dw\Big\} \\
&= \frac{f(a)}{(x-a)^\alpha} + \frac{1}{1-\alpha}\Big\{f'(x)(x-x)^{1-\alpha}\cdot 1 + \int_a^x \frac{f'(w)(1-\alpha)dw}{(x-w)^\alpha}\Big\} \\
&= \frac{f(a)}{(x-a)^\alpha} + \int_a^x \frac{f'(w)dw}{(x-w)^\alpha},
\end{aligned}$$

that is,

$$\frac{d}{dx}\int_a^x \frac{f(w)dw}{(x-w)^\alpha} = \frac{f(a)}{(x-a)^\alpha} + \int_a^x \frac{f'(w)dw}{(x-w)^\alpha}. \tag{2.20}$$

Now, using (2.18), (2.20) and the definition of the Caputo derivative operator for $0 < \alpha < 1$ (2.16), we obtain

$$\mathcal{D}^\alpha_{a^+} f(x) = \frac{f(a)}{\Gamma(1-\alpha)(x-a)^\alpha} +^C \mathcal{D}^\alpha_{a^+} f(x).$$

Thus, (2.17) holds.

□

Example 2.1. *If* $f(z) = \sum_{k=0}^\infty a_k z^k$ *is an analytic function in the unit disk,* $|z| < 1$, *and* $f(0) = 0$, *we have that its Riemann-Liouville fractional derivative and its Caputo fractional derivative of same order* $\alpha > 0$ *satisfy the following relations*

- $\mathcal{D}^\alpha f(z) = z^{-\alpha} \sum_{k=1}^\infty a_k \frac{\Gamma(1+k)}{\Gamma(1+k-\alpha)} z^k$;
- $\mathcal{D}^\alpha f(z) =^C \mathcal{D}^\alpha f(z)$

2.3 Gelfond-Leontiev Derivatives

We now present a third type of fractional derivative, based on power series expansion. Let f be an analytic function in the unit disk $\{|z| < 1\}$ given by

$$f(z) = \sum_{k=0}^\infty a_k\, z^k.$$

In order to generalize the classic derivative of f the natural way is to replace the derivatives of the building blocks z^k by a convenient general multiplier. This idea was the starting point for another concept of fractional derivative introduced by Gelfond and Leontiev in 1951. This concept requires an auxiliar entire function φ which acts as the exponential with respect to this derivative.

Definition 2.7 (GL-derivative, see [4, 39]). *Let*

$$z \mapsto \varphi(z) = \sum_{k=0}^{\infty} \varphi_k z^k, \tag{2.21}$$

be an entire function of order $\rho > 0$ and type $\sigma \neq 0$, that is to say,

$$\lim_{k\to\infty} k^{\frac{1}{\rho}} \sqrt[k]{|\varphi_k|} = (\sigma e \rho)^{\frac{1}{\rho}} .$$

The Gelfond-Leontiev operator of generalized differentiation (GL derivative) with respect to φ is given by

$$f(z) = \sum_{k=0}^{\infty} a_k\, z^k \quad \xrightarrow{\quad D_\varphi \quad} \quad D_\varphi f(z) = \sum_{k=1}^{\infty} a_k \frac{\varphi_{k-1}}{\varphi_k}\, z^{k-1}, \tag{2.22}$$

while the corresponding Gelfond-Leontiev integration operator (GL-integration) is

$$f(z) = \sum_{k=0}^{\infty} a_k\, z^k \quad \xrightarrow{\quad I_\varphi \quad} \quad I_\varphi f(z) = \sum_{k=0}^{\infty} a_k\, \frac{\varphi_{k+1}}{\varphi_k}\, z^{k+1}. \tag{2.23}$$

From the conditions required for φ we have that $\limsup_{k\to\infty} k^{1/\rho} \sqrt[k]{|\varphi_k|} = (\sigma\rho\mathrm{e})^{1/\rho}$. Thus (see [27, 40])

$$\limsup_{k\to\infty} \sqrt[k]{\left|\frac{\varphi_{k-1}}{\varphi_k}\right|} = 1,$$

and, by the Cauchy-Hadamard formula, both series, (2.22) and (2.23), inherit the radius of convergence $R > 0$ of f.

In the following example we present the generalized Gelfond-Leontiev differentiation operator with respect to the Mittag-Leffler function. This entire function was introduced by Dimovski and Kiryakova [15], [16], Kiryakova [22] for real values of the parameter μ and it was extended to a complex parameter μ by Linchouk [23] in 1985.

Example 2.2. *Let φ be the Mittag-Leffler function $E_{\frac{1}{\rho},\mu}$, that is*

$$\varphi(z) = E_{\frac{1}{\rho},\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\left(\mu + \frac{k}{\rho}\right)}, \tag{2.24}$$

for $\rho > 0$ and $\mu \in \mathbb{C}$ such that $\mathrm{Re}(\mu) > 0$. The sequence of coefficients have the general term given by

$$\varphi_k = \frac{1}{\Gamma\left(\mu + \frac{k}{\rho}\right)}, \quad k = 0, 1, 2, \ldots$$

and the GL-derivative and -integration operators, (2.22) and (2.23)) resp., become the so-called Dzrbashjan-Gelfond-Leontiev (DGL) operators of differentiation and integration

$$D_{\rho,\mu}f(z) = \sum_{k=1}^{\infty} a_k \frac{\Gamma\left(\mu + \frac{k}{\rho}\right)}{\Gamma\left(\mu + \frac{k-1}{\rho}\right)} z^{k-1}, \qquad I_{\rho,\mu}f(z) = \sum_{k=0}^{\infty} a_k \frac{\Gamma\left(\mu + \frac{k}{\rho}\right)}{\Gamma\left(\mu + \frac{k+1}{\rho}\right)} z^{k+1}. \tag{2.25}$$

We remark that these operators have being intensively studied in [16, 15, 39].

Example 2.3. *As a second example, we give concrete values for ρ and μ in the DGL operators, and we prove that the resulting operators coincide with the classical derivative and integral operators. Indeed, for $\varphi = E_{1,1}$, the Mittag-Leffler function (2.24) with $\mu = 1$ and $\rho = 1$, we obtain*

$$\varphi(z) = E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z \tag{2.26}$$

and then,

$$D_{1,1}f(z) = \sum_{k=1}^{\infty} a_k \frac{\Gamma(1+k)}{\Gamma(1+(k-1))} z^{k-1} = \sum_{k=1}^{\infty} a_k \frac{k!}{(k-1)!} z^{k-1} = \sum_{k=1}^{\infty} a_k k\, z^{k-1} = \frac{d}{dz} f(z)$$

with

$$I_{1,1}f(z) = \sum_{k=0}^{\infty} a_k \frac{\Gamma(1+k)}{\Gamma(1+(k+1))} z^{k+1} = \sum_{k=1}^{\infty} a_k \frac{k!}{(k+1)!} z^{k+1} = \sum_{k=1}^{\infty} a_k \frac{1}{k+1} z^{k+1} = \int f(z)dz.$$

2.3.1 Relationship between the Gelfond-Leontiev Integration Operators and the Riemann-Liouville Operator

We have seen in the previous subsection that the choice of parameters greatly influences the behaviour of fractional derivatives. We now prove that, for some particular cases, the Gelfond-Leontiev and the Riemann-Liouville integration operators are similar in the sense that there exist transmutation operators which are isomorphisms between the correspondent spaces of functions and they map one operator into the other.

In order to present these transmutation operators, we introduce some preliminary definitions. The first one is the concept of multiple *Erdélyi-Kober (multi-E.-K.) operators.*

Let $\mathcal{C}[0,\infty)$ be the set of real valued functions f with continuous derivatives in $[0,\infty)$ and let $\mathcal{C}_\alpha$ be the linear space of functions

$$C_\alpha := \{f(x) = x^p \widetilde{f}(x); p > \alpha, \widetilde{f} \in \mathcal{C}[0,\infty)\}.$$

Definition 2.8. *Let $m \geq 1$ be an integer, $\beta > 0$, $\gamma_1, \ldots, \gamma_m$ and $\delta_1 > 0, \ldots, \delta_m > 0$ be* *arbitrary real numbers. Consider the ordered set $\gamma = (\gamma_1, \ldots, \gamma_m)$ as a multiweight and $\delta = (\delta_1, \ldots, \delta_m)$ as a (positive) multiorder of integration. For functions $f \in \mathcal{C}_\alpha$, where $\alpha \geq \max_k[-\beta(\gamma_k + 1)]$, we define the multiple Erdélyi-Kober (multi-E.-K.) operators in the following way: for every $f \in \mathcal{C}_\alpha$ let $\mathrm{I}_{\beta,m}^{(\gamma_k),(\delta_k)} f$ be defined as*

$$\mathrm{I}_{\beta,m}^{(\gamma_k),(\delta_k)} f(z) = \int_0^1 G_{m,m}^{m,0}\Big[\sigma| \begin{array}{c} (\gamma_k + \delta_k)_1^m \\ (\delta_k)_1^m \end{array}\Big] f(z\sigma^{\frac{1}{\beta}})d\sigma, \tag{2.27}$$

where $G_{m,m}^{m,0}$ is the special case of the Meijer G-function defined in (2.3). Then, each operator $\mathcal{R}$ acting on f as

$$\mathcal{R}f(z) = z^{\beta\delta_0}\mathrm{I}_{\beta,m}^{(\gamma_k),(\delta_k)} f(z), \tag{2.28}$$

with arbitrary $\delta_0 \geq 0$, is said to be a generalized (m-triple) operator of fractional integration of Riemann-Liouville type. Briefly it is a generalized RL-fractional integral.

For arbitrary $\beta > 0$, $\gamma \in \mathbb{R}$, and $\delta > 0$ (see [39]) the generalized fractional integral $I_{\beta,1}^{\gamma,\delta}$ (2.27) coincides with the well-known Erdélyi-Kober fractional integral, namely:

$$\begin{aligned} \mathrm{I}_\beta^{\gamma,\delta} f(z) &= \int_0^1 \frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)}\sigma^\gamma f(z\sigma^{\frac{1}{\beta}})d\sigma \\ &= z^{-\beta(\gamma+\delta)}\int_0^z \frac{(z^\beta - \tau^\beta)^{\delta-1}}{\Gamma(\delta)}\tau^{\beta\gamma} f(\tau)d(\tau^\beta) \\ &= \mathrm{I}_{\beta,1}^{\gamma,\delta} f(z). \end{aligned} \tag{2.29}$$

Example 2.4. *The Riemann-Liouville (RL) fractional integral of order $\alpha > 0$ defined in (2.4) can be seen as*

$$(I_{0+}^\alpha y)(x) = \frac{1}{\Gamma(\alpha)}\int_0^z (z-\tau)^{(\alpha-1)} f(\tau)d\tau = z^\alpha \mathrm{I}_{1,1}^{0,\alpha} f(z) := \mathcal{R}^\alpha f(z). \tag{2.30}$$

where $\mathcal{R}$ is given by (2.28).

Let us recall the definition of the *generalized Gelfond-Leontiev integration* with respect to the Mittag-Leffler function $E_\rho(z;\mu)$. For $\rho > 0$, $\mu \in \mathbb{C}$, $\Re(\mu) > 0$ and for power series $f(z) = \sum_{k=0}^\infty a_k z^k$ this operator acts in the following way:

$$I_{\rho,\mu} f(z) = \sum_{k=0}^\infty a_k \frac{\Gamma(\mu + \frac{k}{\rho})}{\Gamma(\mu + \frac{k+1}{\rho})} z^{k+1}.$$

Theorem 2.5. *Given f an analytic function in a domain Ω starlike with respect to the origin,* *we obtain that $I_{\rho,\ \mu}$ has the following integral representation*

$$I_{\rho,\mu}f(z) = \frac{z}{\Gamma\left(\frac{1}{\rho}\right)}\int_0^1 (1-\sigma)^{\frac{1}{\rho}-1}\sigma^{\mu-1}f(z\sigma^{\frac{1}{\rho}})d\sigma = z\mathrm{I}_{\rho,1}^{\mu-1,\frac{1}{\rho}}f(z). \tag{2.31}$$

Proof. Since $f(z) = \sum_{k=0}^{\infty} a_k z^k$, it is sufficient to prove (2.31) for an arbitrary $z^k,\ k \geq 0$. Then, we have

$$\begin{aligned} I_{\rho,\mu}(z^k) &= \frac{z}{\Gamma(\frac{1}{\rho})}\int_0^1 (1-\sigma)^{\frac{1}{\rho}-1}\sigma^{\mu-1}\left(z\sigma^{\frac{1}{\rho}}\right)^k d\sigma \\ &= \frac{z^{k+1}}{\Gamma(\frac{1}{\rho})}\int_0^1 (1-\sigma)^{\frac{1}{\rho}-1}\sigma^{\mu+\frac{k}{\rho}-1}d\sigma \\ &= \frac{\Gamma(\mu+\frac{k}{\rho})}{\Gamma(\mu+\frac{k+1}{\rho})} \end{aligned}$$

so that

$$\begin{aligned} I_{\rho,\mu}f(z) &= \sum_{k=0}^{\infty} a_k I_{\rho,\mu}(z^k) \\ &= \sum_{k=0}^{\infty} a_k \frac{\Gamma(\mu+\frac{k}{\rho})}{\Gamma(\mu+\frac{k+1}{\rho})}z^{k+1} \end{aligned}$$

as we desired. □

The previous theorem means that $I_{\rho,\ \mu}$ is, for functions analytic in a domain starlike Ω, a special case of the Erdélyi-Kober fractional integration operator (2.29).

The following theorem presents two transmutation operators, the first relating the DGL generalized integration operators $I_{\rho,1}$, and $I_{\rho,\mu}$, while the second, Ξ^{-1}, relates the Riemann-Liouville (R-L) fractional integral $R^{\frac{1}{\rho}}$ with $I_{\rho,1}$. For more details on these operators, we refer to [39, 17].

Let $\mathcal{A}$ be the space of analytic complex-valued functions f in a disk $|z| < R,\ R > 0$.

Theorem 2.6. *The fractional integration operator*

$$\Phi f(z) = \mathrm{I}_{\rho}^{0,\mu-1}f(z) = \int_0^1 \frac{(1-\sigma)^{\mu-2}}{\Gamma(\mu-1)}f(z\sigma^{\frac{1}{\rho}})d\sigma,\quad \mu \geq 1 \tag{2.32}$$

is a transmutation from the Gelfond-Leontiev (GL) integration operator $I_{\rho,1}$ to the more gen-

eral Dzrbashjan-Gelfond-Leontiev (DGL) operator $I_{\rho,\ \mu}$ *with* $\mu \geq 1$*:*

$$\Phi : I_{\rho,1} \to I_{\rho,\mu}, \ \ i.e. \ \ \Phi L_{\rho,1} = I_{\rho,\mu}\Phi \ \ in \ \ \mathcal{A}.$$

On the other hand, the mapping $\Xi^{-1} : f(z) \to f(z^\rho)$ $(\rho > 0)$ *is a transmutation from the Riemann-Liouville (RL) operator* $\mathcal{R}^{\frac{1}{\rho}}$ *to the operator* $I_{\rho,1}$*:*

$$\Xi^{-1} : \mathcal{R}^{\frac{1}{\rho}} \to I_{\rho,1}, \ \ i.e. \ \ \Xi^{-1}\mathcal{R}^{\frac{1}{\rho}} = I_{\rho,1}\Xi^{-1} \ \ in \ \ \mathcal{A}.$$

Then, the composition $\Phi\Xi^{-1}$*, that is, the integration operator*

$$\Psi f(z) : \mathrm{I}_{\rho}^{0,\mu-1}\Xi^{-1}f(z) = \int_0^1 \frac{(1-\sigma)^{\mu-2}}{\Gamma(\mu-1)} f(z^\rho \sigma) d\sigma \tag{2.33}$$

is a transmutation operator from $\mathcal{R}^{\frac{1}{\rho}}$ *to* $I_{\rho,\ \mu}$*, namely:*

$$\Psi : \mathcal{R}^{\frac{1}{\rho}} \to I_{\rho,\mu}, \ \ or \ \ \Psi\mathcal{R}^{\frac{1}{\rho}} = I_{\rho,\mu}\Psi \ \ in \ \ \mathcal{A}.$$

For the proof we refer to [39].

We observe that the operators $\Phi : I_{\rho,1} \to I_{\rho,\ \mu}$ and $\Psi : \mathcal{R}^{\frac{1}{\rho}} \to I_{\rho,\ \mu}$ admit another representation given as

$$\Phi f(z) = \rho \frac{z^{\rho(\mu-1)}}{\Gamma(\mu-1)} \int_0^z (z^\rho - \zeta^\rho)^{\mu-2} \zeta^{\rho-1} f(z) d\zeta,$$

and

$$\Psi f(z) = \frac{z^{\rho(\mu-1)}}{\Gamma(\mu-1)} \int_0^{z^\rho} (z^\rho - \zeta)^{\mu-2} f(\zeta) d\zeta.$$

These transmutation operators (which can be expressed as fractional integration operators) allow to transfer known results for one given operator to another. In other words, and under some specific conditions, the same result holds for all three operators, GL, DGL, and RL.

2.3.2 Generalized Eigenfunctions of the Fractional Laplace Operator

From Definition (2.7) for the GL-derivatives we conclude that the chosen entire function φ acts as exponential function for the Gelfond-Leontiev operator of generalized differentiation D_φ since it holds

$$D_\varphi \varphi(z) = \sum_{k=1}^{\infty} \varphi_k \frac{\varphi_{k-1}}{\varphi_k} z^{k-1} = \sum_{k=1}^{\infty} \varphi_{k-1} z^{k-1} = \sum_{k=0}^{\infty} \varphi_k z^k = \varphi(z), \tag{2.34}$$

for all $z \in \mathbb{C}$.

Furthermore, we obtain that

$$D_{\varphi}^{n}\varphi = \varphi, \quad \text{for all } n \in \mathbb{N}. \tag{2.35}$$

In particular, we can address the generalized eigenfunctions of the fractional Laplace operator $\Delta^{\alpha} := D_{\varphi}^{2}$, where D_{φ} is the GL-derivative operator. To this effect, we first observe that

$$\Delta^{\alpha}\varphi(\lambda z) = \lambda^2 \varphi(\lambda z),\ \ \lambda \in \mathbb{C}$$

that is, $\varphi(\lambda z)$ is a solution of the fractional Schödinger equation given as

$$(\Delta^{\alpha} - \lambda^2)u = 0.$$

Moreover, by taking an arbitrary $f_{\lambda}(z) = \sum_{k=0}^{\infty} a_k z^k$ such that $\Delta^{\alpha} f_{\lambda} = \lambda^2 f_{\lambda}$ then, from the definition of GL-derivatives, we obtain that

$$\Delta^{\alpha} f_{\lambda} = \sum_{k=2}^{\infty} a_k \frac{\varphi_{k-2}}{\varphi_k} z^{k-2} = \sum_{k=0}^{\infty} a_{k+2} \frac{\varphi_k}{\varphi_{k+2}} z^k. \tag{2.36}$$

and thus,

$$\Delta^{\alpha} f_{\lambda} = \lambda^2 f_{\lambda} \Leftrightarrow \sum_{k=0}^{\infty} a_{k+2} \frac{\varphi_k}{\varphi_{k+2}} z^k = \lambda^2 \sum_{k=0}^{\infty} a_k z^k.$$

It follows that

$$a_{k+2} = \lambda^2 \frac{\varphi_{k+2}}{\varphi_k} a_k \tag{2.37}$$

that is,

$$\begin{cases} a_{2k} = \lambda^{2k} \frac{\varphi_{2k}}{\varphi_0} a_0, & ; \\ a_{2k+1} = \lambda^{2k} \frac{\varphi_{2k+1}}{\varphi_1} a_1, & . \end{cases}$$

Therefore, the function f is given as

$$f_{\lambda}(z) = \frac{a_0}{\varphi_0} \sum_{k=0}^{\infty} (\lambda^{2k}\varphi_{2k}) z^{2k} + \frac{a_1}{\varphi_1} \sum_{k=0}^{\infty} (\lambda^{2k}\varphi_{2k+1}) z^{2k+1}. \tag{2.38}$$

Lemma 2.7. *Let $\lambda \in \mathbb{C}, \lambda \neq 0$. Then the function f given in (2.38) is an entire function.*

Proof. It is sufficient to prove that

$$\sum_{k=0}^{\infty} \lambda^k \varphi_{2k} z^{2k} \quad \text{and} \quad \sum_{k=0}^{\infty} \lambda^k \varphi_{2k+1} z^{2k+1}$$

are entire functions. To this effect,

$$\lim_{k\to\infty} |\varphi_{2k}\lambda^k|^{\frac{1}{2k}} \quad = \quad |\lambda|^{\frac{1}{2}} \lim_{k\to\infty} |\varphi_{2k}|^{\frac{1}{2k}} = 0,$$

that is, $\displaystyle\sum_{k=0}^{\infty} \lambda^k \varphi_{2k} z^{2k}$ is an entire function. Similarly, we have $\displaystyle\lim_{k\to\infty} |\varphi_{2k+1}\lambda^k|^{\frac{1}{2k+1}} = 0$ so that $\displaystyle\sum_{k=0}^{\infty} \lambda^k \varphi_{2k+1} z^{2k+1}$ is also an entire function.

Therefore, f is an entire function.

$\square$

Theorem 2.8. *Let $\lambda \in \mathbb{C}\backslash\{0\}$. Then the space of solutions of $(\Delta_\alpha - \lambda^2)u = 0$ is given by $\mathcal{L}(\varphi_\lambda^{even}, \varphi_\lambda^{odd})$, where $\varphi_\lambda^{even} = \sum_{k=0}^{\infty}(\lambda^{2k}\varphi_{2k})z^{2k}$ and $\varphi_\lambda^{odd} = \sum_{k=0}^{\infty}(\lambda^{2k}\varphi_{2k+1})z^{2k+1}$.*

Chapter 3

Ternary Algebras and Its Symmetries

In this chapter, we introduce the ternary Clifford algebra and present some of its properties. The definition for conjugation and inversion are given and we define an inner product between elements of this algebra which allows to define the Fischer inner product in this setting. We finalise with the explicit computation of the symmetries for a ternary Clifford algebra associated to a vectorial space of dimension 2.

For an algebraic treatment of this type of generalized Clifford algebras (which includes the case of $d = 3$), we refer the reader to the works of Rausch De Traubenberg, among others (see, e.g., [33, 24, 25]). However, we should stress the fact that these papers do not deal with these generalized Clifford algebras in the scope of $N-$fold factorizations of the Laplacian.

3.1 Motivation and Definition

It is well known that many problems in physics and mathematics requires $SU(2)$-symmetries. For instance, the linearization of the relativistic second order wave equation leads to the Dirac equation, which describes spin-$\frac{1}{2}$ hydrogen electrons. Unfortunately, such imposition of $SU(2)$-symmetries is often too restrictive and there are several other physical problems for which one cannot obtain their solution by such restriction. For example, some problems of the supersymmetry theory (which involves associated particles with spin which differs by a half-integer), or the Calogero-Moser dynamical system of $N-$body problem (N equal particles with a harmonic potential), etc. To treat such problems a higher order of symmetry is required. This motivates the analysis of N-fold factorizations of second order operators, and subsequently, of generalized Clifford Algebras.

We now proceed with the definition of *ternary Clifford algebras.*

Definition 3.1. *(Ternary Clifford algebra) Let* $\{e_1, \cdots, e_d\}$ *be a basis of the complex Eu-*

clidean vector space $\mathbb{C}^d$ *(seen as a complexification of the Euclidean vector space in* $\mathbb{R}^d$*). We define* $\mathcal{C}\ell_d^{1/3}$ *as the free algebra generated by* $\mathbb{C}^d$ *subject to the multiplication rule:*

$$[e_i, e_j, e_k] : \;=\; e_ie_je_k + e_ie_ke_j + e_je_ie_k + e_je_ke_i + e_ke_ie_j + e_ke_je_i = 6\delta_{ijk} \qquad (3.1)$$

where the ternary form $[e_i, e_j, e_k]$ *is an extension of the anti-commutator relation for* $i, j, k = 1, \ldots, d$*. Moreover,* $\mathcal{C}\ell_d^{1/3}$ *is called the associated ternary Clifford algebra.*

Lemma 3.1. *Let* $\mathcal{C}\ell_d^{1/3}$ *be a ternary Clifford algebra. Then, the basis elements* $\{e_1, \cdots, e_d\}$ *satisfy*

$$e_ie_j \;=\; \omega e_je_i, \text{ for } 1 \leq i < j \leq d, \qquad (3.2)$$

where ω *is a third root of the unity. In what follows we adopt* $\omega = e^{i2\pi/3}$*.*

Proof. Let $\omega \in \mathbb{C}$ such that

$$e_ie_j = \omega \; e_je_i, \text{ for } 1 \leq i < j \leq d. \qquad (3.3)$$

Then, $e_je_i = \omega^{-1}e_ie_j$. Now, from (3.1) we get

$$e_ie_j + e_je_i + e_i^2e_je_i^2 = 0$$

and by using (3.3), it follows

$$(1 + \omega^{-1} + \omega)e_ie_j = 0.$$

Since $\omega = e^{i2\pi/3}$ and $\omega = e^{i4\pi/3}$ are the only nonzero solutions for $1 + \omega^{-1} + \omega = 0$ and they are also the inverse of each other, we take $\omega = e^{i2\pi/3}$.

□

Let $\boldsymbol{\nu} = (\nu_1, \cdots, \nu_d)$ be an ordered $d-$tuple. A canonical basis for $\mathcal{C}\ell_d^{1/3}$ is given by the set of all ordered products:

$$e^{\boldsymbol{\nu}} := e_1^{\nu_1} \cdots e_d^{\nu_d}. \qquad (3.4)$$

Moreover, and due to the fact that $e_j^3 = 1$ (an immediate consequence of (3.1)) we can restrict ourselves to $\nu_j = 0, 1, 2$, the equivalence classes of congruence modulus 3.

In view of these relations one obtains the following commutator relation:

$$e_i^{\nu_i}e_j^{\mu_j} = \omega^{\nu_i\mu_j} \; e_j^{\mu_j}e_i^{\nu_i}, \quad e_j^{\nu_j}e_i^{\mu_i} = \omega^{2\nu_j\mu_i} \; e_i^{\mu_i}e_j^{\nu_j}, \qquad 1 \leq i < j \leq d, \qquad (3.5)$$

which leads to the multiplication rule between the elements of the basis (3.4) as:

$$e^{\boldsymbol{\nu}}e^{\boldsymbol{\mu}} = \omega^{\boldsymbol{\nu}*\boldsymbol{\mu}} \; e^{\boldsymbol{\nu}+\boldsymbol{\mu}}, \qquad (3.6)$$

with

$$\boldsymbol{\nu} * \boldsymbol{\mu} := 2(\nu_d + \nu_{d-1} + \cdots + \nu_2)\mu_1 + 2(\nu_d + \nu_{d-1} + \cdots + \nu_3)\mu_2 + \cdots + 2\nu_d\mu_{d-1}$$
$$= 2\sum_{j=1}^{d-1}\sum_{s=j+1}^{d} \nu_s\mu_j. \tag{3.7}$$

At this point, we remark that these products have to be understood as products between the correspondent equivalence classes of congruence modulus 3, that is

$$e_j^1 e_j^2 = e_j^3 = e_j^0 = 1 \quad (\nu_j = 1, \mu_j = 2 \quad \Rightarrow \quad \nu_j\mu_j = 0),$$

or

$$e_j^2 e_j^2 = e_j^4 = e_j, \quad (\nu_j = 2, \mu_j = 2 \quad \Rightarrow \quad \nu_j\mu_j = 1).$$

Thus, a ternary Clifford number $a \in \mathcal{C}\ell_d^{1/3}$ can be written as

$$a = \sum_{\boldsymbol{\nu}} a_{\boldsymbol{\nu}}\, e^{\boldsymbol{\nu}},$$

with $a_{\boldsymbol{\nu}} \in \mathbb{C}$, and the ternary Clifford algebra $\mathcal{C}\ell_d^{1/3}$ assumes the form:

$$\mathcal{C}\ell_d^{1/3} = \left\{ a = \sum_{\boldsymbol{\nu}} a_{\boldsymbol{\nu}}\, e^{\boldsymbol{\nu}},\ a_{\boldsymbol{\nu}} \in \mathbb{C}, \boldsymbol{\nu} = (\nu_1, \cdots, \nu_d),\, \nu_j = 0, 1, 2 \right\},$$

where we recall the complex scalars $z \in \mathbb{C}$ commute with the basis elements, i.e., $ze^{\boldsymbol{\nu}} = e^{\boldsymbol{\nu}}z$.

Like in standard Clifford algebras, we shall denote by $[\mathcal{C}\ell_d^{1/3}]_k$ the subspace of all $k-$vectors spanned by $e^{\boldsymbol{\nu}},\ |\boldsymbol{\nu}| = k$. In particular, we have

$$[\mathcal{C}\ell_d^{1/3}]_0 = \mathbb{C}, \qquad \text{subspace of scalars;}$$
$$[\mathcal{C}\ell_d^{1/3}]_1 = \Big\{ a = \sum_{j=1}^{d} a_j\, e_j,\ a_j \in \mathbb{C} \Big\}, \qquad \text{subspace of vectors;}$$
$$\vdots$$
$$[\mathcal{C}\ell_d^{1/3}]_{2d} = \Big\{ a = \sum_{\boldsymbol{\nu}\ s.t. |\boldsymbol{\nu}|=2d} a_{\boldsymbol{\nu}}\, e^{\boldsymbol{\nu}},\ a_{\boldsymbol{\nu}} \in \mathbb{C} \Big\}, \qquad \text{subspace of pseudoscalars.}$$

3.2 Conjugation and Inversion in $\mathcal{C}\ell_d^{1/3}$

Now, we introduce the definition for conjugation in this algebra.

Definition 3.2 (The conjugation). *The conjugation in the ternary Clifford algebra $\mathcal{C}\ell_d^{1/3}$ is defined as the involutory automorphism $\bar{\cdot} : \mathcal{C}\ell_d^{1/3} \to \mathcal{C}\ell_d^{1/3}$ given by:*

$$a \mapsto \overline{a} = \sum_{\boldsymbol{\nu}} \overline{a}_{\boldsymbol{\nu}} \; \overline{e^{\boldsymbol{\nu}}}, \tag{3.8}$$

where $\overline{a}_{\boldsymbol{\nu}}$ denotes the usual complex conjugation and its action on the basis elements $e^{\boldsymbol{\nu}}$ is defined by:

$$\overline{1} = 1, \quad \overline{e_j^{\nu_j}} = e_j^{3-\nu_j}, \quad \nu_j = 0,1,2, \; j = 1, \cdots, d. \tag{3.9}$$

together with

$$\overline{(ua)} = \overline{a}\;\overline{u}, \quad u, a \in \mathcal{C}\ell_d^{1/3}, \tag{3.10}$$

Therefore, we get that

$$\overline{e^{\boldsymbol{\nu}}} = \overline{e_1^{\nu_1} \cdots e_d^{\nu_d}} := e_d^{3-\nu_d} \dots e_1^{3-\nu_1}.$$

Lemma 3.2. *For all $\boldsymbol{\nu} \in \{0,1,2\}^d$ we have*

$$e^{\boldsymbol{\nu}} \overline{e^{\boldsymbol{\nu}}} = \overline{e^{\boldsymbol{\nu}}} e^{\boldsymbol{\nu}} = 1.$$

Moreover, this element $\overline{e^{\boldsymbol{\nu}}}$ can again be expressed in terms of the chosen basis elements (3.4) as:

$$\overline{e^{\boldsymbol{\nu}}} = e_d^{3-\nu_d} \dots e_1^{3-\nu_1} = \omega^{\boldsymbol{\nu}^*} e^{3-\boldsymbol{\nu}}, \tag{3.11}$$

where $3 - \boldsymbol{\nu} := (3-\nu_1, \cdots, 3-\nu_d)$ and $\boldsymbol{\nu}^* := 2\sum_{j=1}^{d-1}\sum_{s=j+1}^{d}(3-\nu_j)(3-\nu_s)$ so that

$$\omega^{\boldsymbol{\nu}^*} := \omega^{2\sum_{j=1}^{d-1}\sum_{s=j+1}^{d}\nu_j\nu_s} = \omega^{2[\nu_1(\nu_2+\cdots+\nu_d)+\nu_2(\nu_3+\cdots+\nu_d)+\cdots+\nu_{d-1}\nu_d]}. \tag{3.12}$$

Again, we remind the reader that the products in (3.11) and (3.12) are to be understood as elements of classes of congruence modulus 3.

We now address the problem of invertibility of vectors in $\mathcal{C}\ell_d^{1/3}$.

Lemma 3.3. *For every vector $a = a_1e_1 + \cdots + a_de_d$ in $\mathcal{C}\ell_d^{1/3}$ it holds*

$$a^3 = a_1^3 + \cdots + a_d^3 \in \mathbb{C}.$$

Proof. Indeed, from

$$a^2 \;=\; a_1^2e_1^2 + \cdots + a_d^2e_d^2 + \sum_{i<j}^{d} a_ia_j(1+\omega^2)e_ie_j$$

we obtain that

$$\begin{aligned} a^3 &= a_1^3 + \cdots + a_d^3 + \sum_{i<j}^{d} a_i^2 a_j (1+\omega+\omega^2) e_i^2 e_j + \sum_{i<j}^{d} a_i a_j^2 (1+\omega+\omega^2) e_i e_j^2 \\ &= a_1^3 + \cdots + a_d^3 \end{aligned}$$

since we have that $1+\omega+\omega^2 = 0$. $\square$

Based on the previous lemma we obtain conditions under which a vector in $\mathcal{C}\ell_d^{1/3}$ has an inverse.

Lemma 3.4. *Let $a = a_1e_1 + \cdots + a_de_d$ be an arbitrary vector in $\mathcal{C}\ell_d^{1/3}$. We say that a is invertible if $a^3 \neq 0$; in this case, its inverse element is given by*

$$a^{-1} = \left(\frac{1}{a^3}\right) a^2. \tag{3.13}$$

Usually, the inverse of an element in a classic Clifford algebra is construct via its conjugate. We now look into conditions under which such an inverse of a general element of $\mathcal{C}\ell_d^{1/3}$ exists.

Theorem 3.5. *Let $a = \sum_{\boldsymbol{\nu}} a_{\boldsymbol{\nu}}\ e^{\boldsymbol{\nu}}$ be an arbitrary element in $\mathcal{C}\ell_d^{1/3}$. Then,*

1. *$a\overline{a} \in \mathbb{C}$ if and only if*
$$\sum_{\boldsymbol{\nu}\neq\boldsymbol{\mu}} \omega^{\boldsymbol{\mu}^*+\boldsymbol{\nu}*(3-\boldsymbol{\mu})}\ a_{\boldsymbol{\nu}}\overline{a}_{\boldsymbol{\mu}} e^{\boldsymbol{\nu}-\boldsymbol{\mu}} = 0,$$
where
$$\omega^{\boldsymbol{\mu}^*+\boldsymbol{\nu}*(3-\boldsymbol{\mu})} = \omega^{2[\sum_{j=1}^{d-1}\mu_j \sum_{s=j+1}^{d}(\mu_s-\nu_s)]}. \tag{3.14}$$

2. *$\overline{a}a \in \mathbb{C}$ if and only if*
$$\sum_{\boldsymbol{\nu}\neq\boldsymbol{\mu}} \omega^{\boldsymbol{\mu}^*+(3-\boldsymbol{\mu})*\boldsymbol{\nu}}\ a_{\boldsymbol{\nu}}\overline{a}_{\boldsymbol{\mu}} e^{\boldsymbol{\nu}-\boldsymbol{\mu}} = 0,$$
where
$$\omega^{\boldsymbol{\mu}^*+(3-\boldsymbol{\mu})*\boldsymbol{\nu}} = \omega^{2[\sum_{j=1}^{d-1}(\mu_j-\nu_j) \sum_{s=j+1}^{d}\mu_s]}. \tag{3.15}$$

3. *if $|a|^2 = a\overline{a} = \overline{a}a \in \mathbb{C}\setminus\{0\}$, then the element a is invertible and its inverse a^{-1} is*
$$a^{-1} = \frac{1}{|a|^2}\overline{a}. \tag{3.16}$$

Proof. We begin by analysing the product of two arbitrary elements $a = \sum_{\boldsymbol{\nu}} a_{\boldsymbol{\nu}}\ e^{\boldsymbol{\nu}}$, $b =$

$\sum_{\boldsymbol{\mu}} b_{\boldsymbol{\mu}}\ e^{\boldsymbol{\mu}} \in \mathcal{C}\ell_d^{1/3}$. Then

$$\begin{aligned} a\bar{b} &= \left(\sum_{\boldsymbol{\nu}} a_{\boldsymbol{\nu}}\ e^{\boldsymbol{\nu}}\right)\left(\sum_{\boldsymbol{\mu}} \bar{b}_{\boldsymbol{\mu}}\ \overline{e^{\boldsymbol{\mu}}}\right) = \sum_{\boldsymbol{\nu},\boldsymbol{\mu}} a_{\boldsymbol{\nu}}\bar{b}_{\boldsymbol{\mu}}\ e^{\boldsymbol{\nu}}\overline{e^{\boldsymbol{\mu}}} \\ &= \sum_{\boldsymbol{\nu}} a_{\boldsymbol{\nu}}\bar{b}_{\boldsymbol{\nu}} e^{\boldsymbol{\nu}}\overline{e^{\boldsymbol{\nu}}} + \sum_{\boldsymbol{\nu}\neq\boldsymbol{\mu}} \omega^{\boldsymbol{\mu}^*} a_{\boldsymbol{\nu}}\bar{b}_{\boldsymbol{\mu}} e^{\boldsymbol{\nu}} e^{3-\boldsymbol{\mu}} \\ &= \sum_{\boldsymbol{\nu}} a_{\boldsymbol{\nu}}\bar{b}_{\boldsymbol{\nu}} + \sum_{\boldsymbol{\nu}\neq\boldsymbol{\mu}} \omega^{\boldsymbol{\mu}^*}\omega^{\boldsymbol{\nu}*(3-\boldsymbol{\mu})}\ a_{\boldsymbol{\nu}}\bar{b}_{\boldsymbol{\mu}} e^{\boldsymbol{\nu}-\boldsymbol{\mu}} \end{aligned} \tag{3.17}$$

where $\omega^{\boldsymbol{\mu}^*}\omega^{\boldsymbol{\nu}*(3-\boldsymbol{\mu})}$ is given by (3.12) and (3.7). Easy computations lead to the explicit formula

$$\omega^{\boldsymbol{\mu}^*+\boldsymbol{\nu}*(3-\boldsymbol{\mu})} = \omega^{2[\sum_{j=1}^{d-1}\mu_j\sum_{s=j+1}^{d}(\mu_s-\nu_s)]}.$$

In particular, we have for $a = \sum_{\boldsymbol{\nu}} a_{\boldsymbol{\nu}}\ e^{\boldsymbol{\nu}}$ that

$$a\bar{a} = \sum_{\boldsymbol{\nu}} |a_{\boldsymbol{\nu}}|^2 + \sum_{\boldsymbol{\nu}\neq\boldsymbol{\mu}} \omega^{\boldsymbol{\mu}^*+\boldsymbol{\nu}*(3-\boldsymbol{\mu})}\ a_{\boldsymbol{\nu}}\bar{a}_{\boldsymbol{\mu}} e^{\boldsymbol{\nu}-\boldsymbol{\mu}},$$

and, thus, the first statement holds.

In a similar way,

$$\bar{a}a = \sum_{\boldsymbol{\nu}} |a_{\boldsymbol{\nu}}|^2 + \sum_{\boldsymbol{\nu}\neq\boldsymbol{\mu}} \omega^{\boldsymbol{\mu}^*+(3-\boldsymbol{\mu})*\boldsymbol{\nu}}\ a_{\boldsymbol{\nu}}\bar{a}_{\boldsymbol{\mu}} e^{\boldsymbol{\nu}-\boldsymbol{\mu}},$$

with $\omega^{\boldsymbol{\mu}^*+(3-\boldsymbol{\mu})*\boldsymbol{\nu}} = \omega^{2[\sum_{j=1}^{d-1}(\mu_j-\nu_j)\sum_{s=j+1}^{d}\mu_s]}$. This completes the second statement.

Finally, if $\bar{a}a = a\bar{a} = \sum_{\boldsymbol{\nu}} |a_{\boldsymbol{\nu}}|^2 := |a|^2 \in \mathbb{C}\setminus\{0\}$ we obtain $a^{-1} = \frac{1}{|a|^2}\ \bar{a}$. □

In order to appreciate the restrictiveness of Theorem 3.5 we give the following example. If $a = a_1e_1 + \cdots + a_de_d \in \mathcal{C}\ell_d^{1/3}$ is such that $a^3 \neq 0$ then we know (Lemma (3.4)) that a has an inverse $a^{-1} = \frac{1}{a^3}a^2$. However, if a has two or more non-zero components ($a_i \neq 0$) then $a\bar{a} \notin \mathbb{C}$. In fact,

$$\bar{a} \quad = \quad \bar{a}_1e_1^2 + \cdots + \bar{a}_de_d^2$$

so that

$$a\bar{a} \quad = \quad |a_1|^2 + \cdots + |a_d|^2 + \sum_{i<j}^{d} a_i\bar{a}_j e_i e_j^2 + \sum_{i<j}^{d} \bar{a}_i a_j \omega e_i^2 e_j,$$

and thus,

$$\sum_{i<j}^{d} a_i\overline{a}_j e_i e_j^2 + \sum_{i<j}^{d} \overline{a}_i a_j \omega e_i^2 e_j = 0 \quad \text{if and only if} \quad a_1 = \ldots = a_d = 0.$$

We recall that for a non-zero vector $a \in \mathcal{C}\ell_d^{1/3}$, we have ensured the existence of an inverse element under the conditions of Lemma 3.4.

We now introduce a definition of a scalar product in $\mathcal{C}\ell_d^{1/3}$.

Theorem 3.6. *The map* $(\cdot|\cdot) : \mathcal{C}\ell^{1/3} \times \mathcal{C}\ell^{1/3} \to \mathbb{C}$ *given by*

$$(a|b) := Sc(a\bar{b}), \quad a, b \in \mathcal{C}\ell^{1/3} \tag{3.18}$$

induces a (complex) scalar product in $\mathcal{C}\ell_d^{1/3}$.

Proof. Given $a = \sum_{\boldsymbol{\nu}} a_{\boldsymbol{\nu}}\, e^{\boldsymbol{\nu}},\ b = \sum_{\boldsymbol{\mu}} b_{\boldsymbol{\mu}}\, e^{\boldsymbol{\mu}}$ in $\mathcal{C}\ell_d^{1/3}$ we have that the mapping (3.18) becomes

$$(a|b) := Sc(a\bar{b}) = \sum_{\boldsymbol{\nu}} a_{\boldsymbol{\nu}} \bar{b}_{\boldsymbol{\nu}}, \tag{3.19}$$

the scalar part of $a\bar{b}$ given by (3.17). Hence, we obtain in a straightforward way

- $(a|a) \geq 0; \quad (a|a) = 0 \;\Rightarrow\; a = 0;$
- $(\lambda a|b) = \lambda(a|b);$
- $(a|b) = \overline{(b|a)};$
- $(a + a'|b) = (a|b) + (a'|b);$

for all $a, a', b \in \mathcal{C}\ell^{1/3},\ \lambda \in \mathbb{C}$.

$\square$

As an example, consider $d = 2$ and let

$$a = a_{00} + a_{10}e_1 + a_{01}e_2 + a_{11}e_1e_2 + a_{12}e_1e_2^2 + a_{21}e_1^2e_2 + a_{20}e_1^2 + a_{02}e_2^2 + a_{22}e_1^2e_2^2$$

be a non-zero element in $\mathcal{C}\ell_2^{1/3}$. Hence,

$$\overline{a} = \overline{a_{00}} + \overline{a_{10}}e_1^2 + \overline{a_{01}}e_2^2 + \overline{a_{11}}e_2^2e_1^2 + \overline{a_{12}}e_2e_1^2 + \overline{a_{21}}e_2^2e_1 + \overline{a_{20}}e_1 + \overline{a_{02}}e_2 + \overline{a_{22}}e_2e_1$$

and, therefore,

$$\begin{aligned} (a|a) &= Sc(a\overline{a}) \\ &= |a_{00}|^2 + |a_{10}|^2 + |a_{01}|^2 + |a_{11}|^2 + |a_{12}|^2 + |a_{21}|^2 + |a_{20}|^2 + |a_{02}|^2 + |a_{22}|^2 \neq 0. \end{aligned}$$

Based on this scalar product we obtain the fractional Fischer inner product in $\mathcal{C}\ell_d^{1/3}$, which will be used in the next chapter to construct a fractional monogenic function theory in the ternary Clifford algebra setting.

3.3 Symmetries of $\mathcal{C}\ell_2^{1/3}$

In this section we explicitly calculate a subgroup of symmetries of $\mathcal{C}\ell_2^{1/3}$. This practical example on the one hand illustrate the difficulty of operating with ternary algebras, and on the other hand, the restrictiveness of the group of symmetries in such cases.

Lemma 3.7. *Let $a, b \in \mathcal{C}\ell_2^{1/3}$ and $x, y \in [\mathcal{C}\ell_2^{1/3}]_1$ given as*

$$\begin{aligned} a &= a_{00} + a_{10}e_1 + a_{01}e_2 + a_{11}e_1e_2 + a_{12}e_1e_2^2 + a_{21}e_1^2e_2 + a_{20}e_1^2 + a_{02}e_2^2 + a_{22}e_1^2e_2^2 \\ x &= x_1e_1 + x_2e_2 \\ y &= y_1e_1 + y_2e_2 \\ b &= b_{00} + b_{10}e_1 + b_{01}e_2 + b_{11}e_1e_2 + b_{12}e_1e_2^2 + b_{21}e_1^2e_2 + b_{20}e_1^2 + b_{02}e_2^2 + b_{22}e_1^2e_2^2. \end{aligned}$$

We have that the following equation

$$ax = yb \tag{3.20}$$

is equivalent to the linear system,

$$A\,v = 0 \tag{3.21}$$

where A is the matrix $A = [a_{ij}]_{9\times 4}$ and v is the vector $v = [v_j]_{4\times 1}$ with entrances given by

$$A = \begin{pmatrix} a_{20} & a_{02} & -b_{20} & -b_{02} \\ a_{00} & a_{12} & -b_{00} & -\omega^2 b_{12} \\ \omega^2 a_{21} & a_{00} & -b_{21} & -b_{00} \\ \omega^2 a_{01} & a_{10} & -b_{01} & -\omega^2 b_{10} \\ \omega a_{02} & a_{11} & -b_{02} & -\omega^2 b_{11} \\ \omega^2 a_{11} & a_{20} & -b_{11} & -\omega b_{20} \\ \omega a_{12} & a_{21} & -b_{12} & -\omega b_{21} \\ a_{10} & a_{22} & -b_{10} & -\omega b_{22} \\ \omega a_{22} & a_{01} & -b_{22} & -b_{01} \end{pmatrix}, \quad v = \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix}$$

and $0 = [0]_{9\times 1}$ is vector.

Proof. Indeed, from

$$ax = yb \Leftrightarrow 0 = ax - yb,$$

it follows that

$$\begin{aligned}
0 &= ax - yb \\
&= [(a_{20}x_1 + a_{02}x_2) - (b_{20}y_1 + b_{02}y_2)] + [(a_{00}x_1 + a_{12}x_2) - (b_{00}y_1 + \omega^2 b_{12}y_2)]e_1 \\
&\quad +[(\omega^2 a_{21}x_1 + a_{00}x_2) - (b_{21}y_1 + b_{00}y_2)]e_2 + [(\omega^2 a_{01}x_1 + a_{10}x_2) - (b_{01}y_1 + \omega^2 b_{10}y_2)]e_1e_2 \\
&\quad +[(\omega a_{02}x_1 + a_{11}x_2) - (b_{02}y_1 + \omega^2 b_{11}y_2)]e_1e_2^2 + [(\omega^2 a_{11}x_1 + a_{20}x_2) - (b_{11}y_1 + \omega b_{20}y_2)]e_1^2e_2 \\
&\quad +[(\omega a_{12}x_1 + a_{21}x_2) - (b_{12}y_1 + \omega b_{21}y_2)]e_1^2e_2^2 + [(a_{10}x_1 + a_{22}x_2) - (b_{10}y_1 + \omega b_{22}y_2)]e_1^2 \\
&\quad +[(\omega a_{22}x_1 + a_{01}x_2) - (b_{22}y_1 + b_{01}y_2)]e_2^2,
\end{aligned}$$

so that we obtain the matricial form $A\,v = 0$, where A is the v matrix given by

$$A = \begin{pmatrix}
a_{20} & a_{02} & -b_{20} & -b_{02} \\
a_{00} & a_{12} & -b_{00} & -\omega^2 b_{12} \\
\omega^2 a_{21} & a_{00} & -b_{21} & -b_{00} \\
\omega^2 a_{01} & a_{10} & -b_{01} & -\omega^2 b_{10} \\
\omega a_{02} & a_{11} & -b_{02} & -\omega^2 b_{11} \\
\omega^2 a_{11} & a_{20} & -b_{11} & -\omega b_{20} \\
\omega a_{12} & a_{21} & -b_{12} & -\omega b_{21} \\
a_{10} & a_{22} & -b_{10} & -\omega b_{22} \\
\omega a_{22} & a_{01} & -b_{22} & -b_{01}
\end{pmatrix},$$

$v = (x_1, x_2, y_1, y_2)^T$ and $0 = [0]_{9\times 1}$ is zero-vector. □

Obviously, if $\mathrm{Rank}(A) = 4$ then the system $Av = 0$ admits the unique solution $v = 0$, that is, $ax = yb$ implies $x = y = 0$. We now look for conditions under which the system (3.21) has non-trivial solutions.

Lemma 3.8. *For $a, b \in \mathcal{C}\ell_2^{1/3}$ let A denote the 9×4 matrix in (3.21),*

$$A = \begin{pmatrix}
a_{20} & a_{02} & -b_{20} & -b_{02} \\
a_{00} & a_{12} & -b_{00} & -\omega^2 b_{12} \\
\omega^2 a_{21} & a_{00} & -b_{21} & -b_{00} \\
\omega^2 a_{01} & a_{10} & -b_{01} & -\omega^2 b_{10} \\
\omega a_{02} & a_{11} & -b_{02} & -\omega^2 b_{11} \\
\omega^2 a_{11} & a_{20} & -b_{11} & -\omega b_{20} \\
\omega a_{12} & a_{21} & -b_{12} & -\omega b_{21} \\
a_{10} & a_{22} & -b_{10} & -\omega b_{22} \\
\omega a_{22} & a_{01} & -b_{22} & -b_{01}
\end{pmatrix}.$$

If the sub-matrices of A,

$$A_1 = \begin{pmatrix} a_{20} & a_{02} & -b_{20} & -b_{02} \\ \omega a_{02} & a_{11} & -b_{02} & -\omega^2 b_{11} \\ \omega^2 a_{11} & a_{20} & -b_{11} & -\omega b_{20} \end{pmatrix}, \qquad A_2 = \begin{pmatrix} a_{00} & a_{12} & -b_{00} & -\omega^2 b_{12} \\ \omega^2 a_{21} & a_{00} & -b_{21} & -b_{00} \\ \omega a_{12} & a_{21} & -b_{12} & -\omega b_{21} \end{pmatrix},$$

and

$$A_3 = \begin{pmatrix} \omega^2 a_{01} & a_{10} & -b_{01} & -\omega^2 b_{10} \\ a_{10} & a_{22} & -b_{10} & -\omega b_{22} \\ \omega a_{22} & a_{01} & -b_{22} & -b_{01} \end{pmatrix}$$

have rank 1 *and system (3.21) has non-trivial solutions then the elements a, b have the form*

$$\begin{cases} a = a_{00}(1 + \omega e_1^2 e_2 + e_1 e_2^2) + a_{20}(e_1 e_2 + e_1^2 + \omega e_2^2) + a_{22}(\omega e_1 + e_2 + e_1^2 e_2^2) \\ b = b_{00}(1 + \omega e_1 e_2^2 + e_1^2 e_2) + b_{02}(e_1 e_2 + \omega e_1^2 + e_2^2) + b_{22}(e_1 + \omega e_2 + e_1^2 e_2^2). \end{cases}$$

In this case, we say that (a, b) is a ***symmetry pair****.*

Proof. If A_1 has rank 1 then its elements satisfy

$$\begin{cases} \omega a_{02} = \alpha a_{20}, & \omega^2 a_{11} = \beta a_{20}; \\ a_{11} = \alpha a_{02}, & a_{20} = \beta a_{02}; \\ b_{02} = \alpha b_{20}, & b_{11} = \beta b_{20}; \\ \omega^2 b_{11} = \alpha b_{02}, & \omega b_{20} = \beta b_{02} \end{cases},$$

for some α, β. Solving this system, we obtain $\alpha = \beta = \omega^2$. In the same way, we obtain for A_2 that

$$\begin{cases} \omega^2 a_{21} = \lambda a_{00}, & \omega a_{12} = \gamma a_{00}; \\ a_{00} = \lambda a_{12}, & a_{21} = \gamma a_{12}; \\ b_{21} = \lambda b_{00}, & b_{12} = \gamma b_{00}; \\ b_{00} = \lambda \omega^2 b_{12}, & \omega b_{21} = \gamma \omega^2 b_{12}, \end{cases}$$

for some λ, γ. Solving the system leads to $\lambda = 1$ and $\gamma = \omega$. Finally, for A_3 we obtain the system

$$\begin{cases} \omega^2 a_{01} = \theta a_{10}, & \omega a_{22} = \xi a_{10}; \\ a_{10} = \theta a_{22}, & a_{01} = \xi a_{22}; \\ b_{01} = \theta b_{10}, & b_{22} = \xi b_{10}; \\ \omega^2 b_{10} = \theta w b_{22}, & b_{01} = \xi \omega b_{22} \end{cases}$$

for some θ, ξ, and we obtain $\theta = \omega$ and $\xi = 1$.

Combining, we get

$$\begin{aligned} a &= a_{00}(1+\omega e_1^2e_2+e_1e_2^2)+a_{20}(e_1e_2+e_1^2+\omega e_2^2)+a_{22}(\omega e_1+e_2+e_1^2e_2^2) \qquad (3.22)\\ b &= b_{00}(1+\omega e_1e_2^2+e_1^2e_2)+b_{02}(e_1e_2+\omega e_1^2+e_2^2)+b_{22}(e_1+\omega e_2+e_1^2e_2^2). \end{aligned}$$

□

We now study the conditions for the existence of the inverse of the element $b \in \mathcal{C}\ell_2^{1/3}$ of a symmetry pair.

Lemma 3.9. *Let $b \in \mathcal{C}\ell_2^{1/3}$ be of the form*

$$b = b_{00}(1+\omega e_1e_2^2+e_1^2e_2)+b_{02}(e_1e_2+\omega e_1^2+e_2^2)+b_{22}(e_1+\omega e_2+e_1^2e_2^2),$$

where $b_{00}, b_{02}, b_{22} \in \mathbb{C}$. Then, there exists the inverse b^{-1} if only if $b_{00} \neq 0$, $b_{02} \neq 0$ and $b_{22} \neq 0$. Moreover, we have b^{-1} given as

$$\begin{aligned} b^{-1} = \frac{1}{9b_{00}}(1+e_1^2e_2+\omega e_1e_2^2)+\frac{1}{9b_{22}}(e_1^2+\omega^2e_2^2+\omega^2e_1e_2)\\ +\frac{1}{9b_{02}}(e_2+\omega^2e_1+\omega^2e_1^2e_2^2). \qquad (3.23) \end{aligned}$$

Proof. Given the element $b = b_{00}(1+we_1e_2^2+e_1^2e_2)+b_{02}(e_1e_2+we_1^2+e_2^2)+b_{22}(e_1+we_2+e_1^2e_2^2)$ we have to find $q \in \mathcal{C}\ell_2^{1/3}$ such that $bq = 1$, that is, $q = b^{-1}$. To this end, let

$$q = q_{00}+q_{10}e_1+q_{01}e_2+q_{11}e_1e_2+q_{12}e_1e_2^2+q_{21}e_1^2e_2+q_{20}e_1^2+q_{02}e_2^2+q_{22}e_1^2e_2^2$$

then, from $bq = 1$, it follows the next linear system

$$\begin{pmatrix} b_{00} & b_{22} & b_{22}\omega & b_{02}\omega & b_{00} & b_{00}\omega^2 & b_{22}\omega & b_{02}\omega & b_{02}\\ b_{22} & b_{02}\omega & b_{02} & b_{00}\omega & b_{22}\omega^2 & b_{22} & b_{02}\omega & b_{00} & b_{00}\omega\\ b_{22}\omega & b_{02}\omega & b_{02} & b_{00} & b_{22} & b_{22}\omega & b_{02}\omega & b_{00}\omega^2 & b_{00}\\ b_{02} & b_{00}\omega & b_{00}\omega & b_{22}\omega^2 & b_{02}\omega & b_{02}\omega & b_{00} & b_{22} & b_{22}\\ b_{00}\omega & b_{22}\omega^2 & b_{22} & b_{02}\omega & b_{00}\omega & b_{00} & b_{22} & b_{02}\omega & b_{02}\\ b_{00} & b_{22}\omega^2 & b_{22} & b_{02}\omega^2 & b_{00} & b_{00}\omega^2 & b_{22} & b_{02}\omega^2 & b_{02}\omega\\ b_{22} & b_{02}\omega^2 & b_{02}\omega & b_{00} & b_{22}\omega^2 & b_{22} & b_{02}\omega^2 & b_{00}\omega^2 & b_{00}\\ b_{02}\omega & b_{00} & b_{00} & b_{22}\omega^2 & b_{02}\omega^2 & b_{02}\omega^2 & b_{00}\omega^2 & b_{22} & b_{22}\\ b_{02} & b_{00} & b_{00} & b_{22} & b_{02}\omega & b_{02}\omega & b_{00}\omega^2 & b_{22}\omega & b_{22}\omega \end{pmatrix} \begin{pmatrix} q_{00}\\ q_{20}\\ q_{02}\\ q_{22}\\ q_{21}\\ q_{12}\\ q_{11}\\ q_{10}\\ q_{01} \end{pmatrix} = \begin{pmatrix} 1\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}.$$

Now, if

$$B = \begin{pmatrix} b_{00} & b_{22} & b_{22}\omega & b_{02}\omega & b_{00} & b_{00}\omega^2 & b_{22}\omega & b_{02}\omega & b_{02} \\ b_{22} & b_{02}\omega & b_{02} & b_{00}\omega & b_{22}\omega^2 & b_{22} & b_{02}\omega & b_{00} & b_{00}\omega \\ b_{22}\omega & b_{02}\omega & b_{02} & b_{00} & b_{22} & b_{22}\omega & b_{02}\omega & b_{00}\omega^2 & b_{00} \\ b_{02} & b_{00}\omega & b_{00}\omega & b_{22}\omega^2 & b_{02}\omega & b_{02}\omega & b_{00} & b_{22} & b_{22} \\ b_{00}\omega & b_{22}\omega^2 & b_{22} & b_{02}\omega & b_{00}\omega & b_{00} & b_{22} & b_{02}\omega & b_{02} \\ b_{00} & b_{22}\omega^2 & b_{22} & b_{02}\omega^2 & b_{00} & b_{00}\omega^2 & b_{22} & b_{02}\omega^2 & b_{02}\omega \\ b_{22} & b_{02}\omega^2 & b_{02}\omega & b_{00} & b_{22}\omega^2 & b_{22} & b_{02}\omega^2 & b_{00}\omega^2 & b_{00} \\ b_{02}\omega & b_{00} & b_{00} & b_{22}\omega^2 & b_{02}\omega^2 & b_{02}\omega^2 & b_{00}\omega^2 & b_{22} & b_{22} \\ b_{02} & b_{00} & b_{00} & b_{22} & b_{02}\omega & b_{02}\omega & b_{00}\omega^2 & b_{22}\omega & b_{22}\omega \end{pmatrix}$$

then

$$\det(B) = -19683(b_{00})^3(b_{02})^3(b_{22})^3$$

so that its determinant is nonzero if only if $b_{00} \neq 0$, $b_{02} \neq 0$ and $b_{22} \neq 0$. Moreover, from the inverse

$$B^{-1} = \frac{1}{9} \begin{pmatrix} \frac{1}{b_{00}} & \frac{1}{b_{22}} & \frac{\omega^2}{b_{22}} & \frac{1}{b_{02}} & \frac{\omega^2}{b_{00}} & \frac{1}{b_{00}} & \frac{1}{b_{22}} & \frac{\omega^2}{b_{02}} & \frac{1}{b_{02}} \\ \frac{1}{b_{22}} & \frac{\omega^2}{b_{02}} & \frac{\omega^2}{b_{02}} & \frac{\omega^2}{b_{00}} & \frac{\omega}{b_{22}} & \frac{\omega}{b_{22}} & \frac{\omega}{b_{02}} & \frac{1}{b_{00}} & \frac{1}{b_{00}} \\ \frac{\omega^2}{b_{22}} & \frac{1}{b_{02}} & \frac{1}{b_{02}} & \frac{\omega^2}{b_{00}} & \frac{1}{b_{22}} & \frac{1}{b_{22}} & \frac{\omega^2}{b_{02}} & \frac{1}{b_{00}} & \frac{1}{b_{00}} \\ \frac{\omega^2}{b_{02}} & \frac{\omega^2}{b_{00}} & \frac{1}{b_{00}} & \frac{\omega}{b_{22}} & \frac{\omega^2}{b_{02}} & \frac{\omega}{b_{02}} & \frac{1}{b_{00}} & \frac{\omega}{b_{22}} & \frac{1}{b_{22}} \\ \frac{1}{b_{00}} & \frac{\omega}{b_{22}} & \frac{1}{b_{22}} & \frac{\omega^2}{b_{02}} & \frac{\omega^2}{b_{00}} & \frac{1}{b_{00}} & \frac{\omega}{b_{22}} & \frac{\omega}{b_{02}} & \frac{\omega^2}{b_{02}} \\ \frac{\omega}{b_{00}} & \frac{1}{b_{22}} & \frac{\omega^2}{b_{22}} & \frac{\omega^2}{b_{20}} & \frac{1}{b_{00}} & \frac{\omega}{b_{00}} & \frac{1}{b_{22}} & \frac{\omega}{b_{02}} & \frac{\omega^2}{b_{02}} \\ \frac{\omega^2}{b_{22}} & \frac{\omega^2}{b_{02}} & \frac{\omega^2}{b_{02}} & \frac{1}{b_{00}} & \frac{1}{b_{22}} & \frac{1}{b_{22}} & \frac{\omega}{b_{02}} & \frac{\omega}{b_{00}} & \frac{\omega}{b_{00}} \\ \frac{\omega^2}{b_{02}} & \frac{1}{b_{00}} & \frac{\omega}{b_{00}} & \frac{1}{b_{22}} & \frac{\omega^2}{b_{02}} & \frac{\omega}{b_{02}} & \frac{\omega}{b_{00}} & \frac{1}{b_{22}} & \frac{\omega^2}{b_{22}} \\ \frac{1}{b_{02}} & \frac{\omega^2}{b_{00}} & \frac{1}{b_{00}} & \frac{1}{b_{22}} & \frac{1}{b_{02}} & \frac{\omega^2}{b_{02}} & \frac{1}{b_{00}} & \frac{1}{b_{22}} & \frac{\omega^2}{b_{22}} \end{pmatrix}$$

we get

$$b^{-1} = q = \frac{1}{9b_{00}}(1 + e_1^2 e_2 + we_1 e_2^2) + \frac{1}{9b_{22}}(e_1^2 + w^2 e_2^2 + w^2 e_1 e_2)$$
$$+ \frac{1}{9b_{02}}(e_2 + w^2 e_1 + w^2 e_1^2 e_2^2).$$

□

Theorem 3.10. *Let*

$$a = a_{00}(1 + \omega e_1^2 e_2 + e_1 e_2^2) + a_{20}(e_1 e_2 + e_1^2 + \omega e_2^2) + a_{22}(\omega e_1 + e_2 + e_1^2 e_2^2) \in \mathcal{C}\ell_2^{1/3},$$

with $a_{00}, a_{20}, a_{22} \in \mathbb{C}$. *Then it holds*

1. *it exists* $b \in \mathcal{C}\ell_2^{1/3}$ *such that* (a, b) *is a symmetry pair;*

2. *for all $x \in [\mathcal{C}\ell_2^{1/3}]_1$ there exists $y \in [\mathcal{C}\ell_2^{1/3}]_1$ such that*

$$ax = yb. \tag{3.24}$$

3. *If b^{-1} exists then we have*

$$\frac{a_{00}}{b_{00}} = \frac{a_{20}}{b_{02}} = \frac{a_{22}}{b_{22}}, \tag{3.25}$$

and we get $y_1 = \dfrac{a_{00}}{b_{00}} x_2$ and $y_2 = \dfrac{a_{00}}{b_{00}} x_1$.

Proof. Proposition 1. is a direct consequence of Lemma 3.8, relations (3.22). From these, it follows that

$$\begin{aligned}
ax &= a_{00}[(x_1+x_2)e_1 + (x_1+x_2)e_2 + w(x_1+x_2)e_1^2e_2^2] \\
&\quad + a_{20}[(x_1+wx_2) + (w^2x_1+x_2)e_1^2e_2 + (w^2x_1+x_2)e_1e_2^2] \\
&\quad + a_{22}[(wx_1+x_2)e_1^2 + (wx_1+x_2)e_2^2 + (w^2x_1+wx_2)e_1e_2] \\
yb &= b_{00}[(y_1+y_2)e_1 + (y_1+y_2)e_2 + w(y_1+y_2)e_1^2e_2^2] \\
&\quad + b_{02}[(wy_1+y_2) + (y_1+w^2y_2)e_1^2e_2 + (y_1+w^2y_2)e_1e_2^2] \\
&\quad + b_{22}[(y_1+wy_2)e_1^2 + (y_1+wy_2)e_2^2 + (wy_1+w^2y_2)e_1e_2].
\end{aligned}$$

so that to obtain $ax = yb$ is equivalent to solve the following linear system:

$$\begin{aligned}
b_{00}(y_1+y_2) &= (x_1+x_2)a_{00} \\
b_{02}(wy_1+y_2) &= (x_1+wx_2)a_{20} \\
b_{22}(y_1+wy_2) &= (wx_1+x_2)a_{22}.
\end{aligned} \tag{3.26}$$

Hence, proposition 2. holds.

Finally, if the element $b = b_{00}(1+\omega e_1e_2^2+e_1^2e_2)+b_{02}(e_1e_2+\omega e_1^2+e_2^2)+b_{22}(e_1+\omega e_2+e_1^2e_2^2) \in \mathcal{C}\ell_2^{1/3}$ has inverse, then the system (3.26) is equivalent to

$$\begin{aligned}
\omega^2(y_1+y_2) &= \omega^2(x_1+x_2)\frac{a_{00}}{b_{00}} \\
(\omega y_1+y_2) &= (x_1+\omega x_2)\frac{a_{20}}{b_{02}} \\
(y_1+\omega y_2) &= (\omega x_1+x_2)\frac{a_{22}}{b_{22}}
\end{aligned} \tag{3.27}$$

and summing these equations leads to

$$0 = (\frac{a_{00}}{b_{00}}\omega^2 + \frac{a_{20}}{b_{02}} + \frac{a_{22}}{b_{22}}\omega)x_1 + (\frac{a_{00}}{b_{00}}\omega^2 + \frac{a_{20}}{b_{02}}\omega + \frac{a_{22}}{b_{22}})x_2, \tag{3.28}$$

for arbitrary x_1, $x_2 \in \mathbb{C}$. Hence,

$$\frac{a_{00}}{b_{00}} = \frac{a_{20}}{b_{02}} = \frac{a_{22}}{b_{22}}.$$

Using (3.25) and taking account the product between ω and the second equation in (3.26), we can obtain

$$\begin{aligned} (y_1 + y_2) &= (x_1 + x_2)\frac{a_{00}}{b_{00}} \\ \omega^2(\omega y_1 + y_2) &= \omega^2(x_1 + \omega x_2)\frac{a_{00}}{b_{00}} \\ (y_1 + \omega y_2) &= (\omega x_1 + x_2)\frac{a_{00}}{b_{00}} \end{aligned} \tag{3.29}$$

and after taking the sum of the equations in (3.29), we have

$$y_1 = \frac{a_{00}}{b_{00}} x_2$$

and

$$y_2 = \frac{a_{00}}{b_{00}} x_1.$$

□

We now restrict ourselves to the case $y = x$ in equation (3.20) in order to find a subset of "rotations" for $\mathcal{C}\ell_2^{1/3}$, that is, $axb^{-1} = x$.

Theorem 3.11. *If $y = x$ in the equation (3.20) we have the following three possibilities for x_1 and x_2:*

(a) If $\frac{a_{00}}{b_{00}} = 1$ then $x_1 = x_2$;

(b) If $\frac{a_{00}}{b_{00}} = -1$ then $x_1 = -x_2$;

(c) If $(\frac{a_{00}}{b_{00}})^2 \neq 1$ then $x_1 = 0 = x_2$.

Proof. If $y = x$ and using (3.25) in (3.26), we have

$$\begin{aligned} (x_1 + x_2) &= (x_1 + x_2)\frac{a_{00}}{b_{00}} \\ (\omega x_1 + x_2) &= (x_1 + \omega x_2)\frac{a_{00}}{b_{00}} \\ (x_1 + \omega x_2) &= (\omega x_1 + x_2)\frac{a_{00}}{b_{00}}. \end{aligned} \tag{3.30}$$

Note that $(\omega x_1 + x_2) = (\omega x_1 + x_2)(\frac{a_{00}}{b_{00}})^2$ and it follows

$$(\omega x_1 + x_2)[(\frac{a_{00}}{b_{00}})^2 - 1] = 0. \tag{3.31}$$

Now we have three possibilities for (3.31):

- If $\dfrac{a_{00}}{b_{00}} = 1$, then $\begin{cases} x_1 + x_2 = x_1 + x_2, \\ \omega x_1 + x_2 = x_1 + \omega x_2, \end{cases}$ and it follows $x_1 = x_2$.
- If $\dfrac{a_{00}}{b_{00}} = -1$, then $\begin{cases} x_1 + x_2 = -(x_1 + x_2), \\ \omega x_1 + x_2 = -(x_1 + \omega x_2), \end{cases}$ and it follows $x_1 = -x_2$.
- If $[(\dfrac{a_{00}}{b_{00}})^2 - 1] \neq 0$, then $\begin{cases} x_1 + x_2 = 0, \\ \omega x_1 + x_2 = 0, \\ x_1 + \omega x_2 = 0, \end{cases}$ and it follows $x_1 = 0 = x_2$.

$\square$

According to theorems (3.10) and (3.11), we can prove the next statement.

Theorem 3.12. *If $y = axb^{-1}$ then the three symmetries of $\mathcal{C}\ell_2^{1/3}$ are given as*

(A) $y = x$ (identity)

(B) $y_1 = x_2$ and $y_2 = x_1$ (reflection)

(C) $y_1 = -x_2$ and $y_2 = -x_1$ (rotation and reflection)

Chapter 4

Fractional Homogeneous Monogenic Polynomials - FHMP

In this chapter we construct the function theory for fractional Dirac operators based on Gelfond-Leontiev operators of generalized differentiation. This construction will be done in the case of the classic Clifford algebras and in the ternary setting. We establish the adequate definition of Fischer inner product and the corresponding Fischer decomposition in both cases. In addition, we prove that in the case of ternary algebras we obtain a cubic factorization of the Laplacian and based on this we establish the corresponding function theory.

We also present an algorithm implemented in MATLAB that allows to compute the coefficients of monogenic homogeneous polynomials, that is, the basis elements of the space of fractional homogeneous monogenic polynomials that arise in the three-dimensional case. These results can also be found on [29, 28].

4.1 FHMP with respect to Clifford Algebras

We present the basic tools of a fractional function theory in higher dimensions by means of a fractional correspondence to the Weyl relations via Gelfond-Leontiev operators of generalized differentiation.

We consider an element $x \in \mathbb{C}^d$ defined as

$$x = \sum_{i=1}^{d} x_i^\alpha e_i \quad \in \quad \mathbb{C}^d, \tag{4.1}$$

where each component x_i^α is a fractional power which should be understood as

$$x_i^\alpha = \begin{cases} \exp(\alpha \ln|x_i|); & x_i^\alpha > 0 \\ 0; & x_i^\alpha = 0 \\ \exp(\alpha \ln|x_i| + \mathbf{i}\alpha\pi); & x_i^\alpha < 0 \end{cases}, \tag{4.2}$$

with $0 < \alpha < 1$ and where $\mathbf{i}$ represents the imaginary unit. We remind that we restrict ourselves to the case of $\alpha \in]0,1[$. Indeed, for values of α outside this range one can always reduce to the previous case via $\alpha = [\alpha] + \tilde{\alpha}$, where $[\alpha]$ denotes its integer part and $\tilde{\alpha} \in]0,1[$.

4.1.1 Fractional Sommen-Weyl Relations

In order to establish a function theory in higher dimensions, the standard approach is the construction of the analogues to the Euler and Gamma operators and of the corresponding Sommen-Weyl relations. To this effect, we have to study the commutator and the anti-commutator between x given in (4.1) and $\mathcal{D} = \sum_{i=1}^{d} \mathcal{D}_i^\alpha e_i$, where $\mathcal{D}_i^\alpha$ is the GL differentiation operator with respect to x_i^α given in (4.2). We recall that $\mathcal{D}_i^\alpha$ is the linear operator which acts on powers of $(x_i^\alpha)^n$ as

$$\mathcal{D}_i^\alpha (x_i^\alpha)^n = \begin{cases} 0, & n=0; \\ \frac{\varphi_{n-1}}{\varphi_n}(x_i^\alpha)^{n-1}, & n=1,2,\cdots. \end{cases}$$

where φ is an entire function with order $\rho > 0$ and type $\sigma \neq 0$.

Now, from the definition of the commutator, we obtain the following fractional relations:

Lemma 4.1. *Let $\mathcal{D}_i^\alpha$ be the GL differentiation operator with respect to x_i^α. Then we have*

$$\left[D_i^\alpha, x_j^\alpha \right] (x_r^\alpha)^l \quad = \quad \begin{cases} 0, & \textit{if } i \neq j; \\ \varphi(1,0)\ (x_r^\alpha)^l, & \textit{if } i=j \ \wedge \ i \neq r; \\ \varphi_D(l+1,l-1)\,(x_r^\alpha)^l, & \textit{if } i=j=r, \end{cases} \tag{4.3}$$

with $l \in \mathbb{N}$, $i=1,\ldots,d$, $0<\alpha<1$, $\varphi(l,k) = \dfrac{\varphi_k}{\varphi_l}$, and $\varphi_D(l+1,l-1) = \varphi(l+1,l) - \varphi(l,l-1)$.

Proof. In fact, we have

$$\mathcal{D}_i^\alpha\,(x_i^\alpha)^l = \varphi(l,l-1)\,(x_i^\alpha)^{l-1} \tag{4.4}$$

so that, in the first case of (4.3)

$$\begin{aligned} \left[D_i^\alpha, x_j^\alpha \right] (x_r^\alpha)^l &= \left(D_i^\alpha\, x_j^\alpha - x_j^\alpha\, D_i^\alpha \right) (x_r^\alpha)^l \\ &= x_j^\alpha \left(D_i^\alpha\,(x_r^\alpha)^l - D_i^\alpha\,(x_r^\alpha)^l \right) \\ &= 0 \end{aligned}$$

since $i \neq j$. In the second case, we consider $i=j \ \wedge \ i \neq r$; then

$$\begin{aligned} [\, D_i^\alpha, x_i^\alpha\,]\,(x_r^\alpha)^l &= \left(D_i^\alpha\, x_i^\alpha\,(x_r^\alpha)^l - x_i^\alpha\, D_i^\alpha\,(x_r^\alpha)^l \right) \\ &= (x_r^\alpha)^l\,(D_i^\alpha\, x_i^\alpha) \\ &= \varphi(1,0)\ (x_r^\alpha)^l \end{aligned}$$

Finally, for $i=j=r$ we obtain

$$
\begin{aligned}
[\,D_i^\alpha, x_i^\alpha\,]\;(x_i^\alpha)^l &= \left(D_i^\alpha\;(x_i^\alpha)^{l+1} - x_i^\alpha\, D_i^\alpha\,(x_i^\alpha)^l\right) \\
&= \varphi(l+1,l) - \varphi(l,l-1) \\
&= \varphi_D(l+1,l-1)\,(x_r^\alpha)^l\,.
\end{aligned}
$$

□

Example 4.1. *If $\varphi(\lambda)$ is the Mittag-Leffler function with $\mu = 1$ and $\dfrac{1}{\rho} = \alpha$ (see Example 2.2) we have $\varphi_k = \dfrac{1}{\Gamma\left(1+k\alpha\right)}$ so that*

$$
\varphi(a,b) = \frac{\Gamma(a\alpha+1)}{\Gamma(b\alpha+1)}.
$$

Theorem 4.2. *The Sommen-Weyl relations for x, $\mathcal{D}$ and $(\underline{x}^\alpha)^{\underline{l}} = \prod_{i=1}^d (x_i^\alpha)^{l_i}$, with $\underline{l} = (l_1,\ldots,l_d)$, and $l = |\underline{l}| = l_1+\ldots+l_d$, are given by:*

$$
\begin{aligned}
\{\mathcal{D},x\}\;(\underline{x}^\alpha)^{\underline{l}} &= -\left(\sum_{r=1}^d \varphi_D(l_r+1,l_r-1) + 2\mathbb{E}^\alpha\right)\;(\underline{x}^\alpha)^{\underline{l}} \\
[\mathcal{D},x]\;(\underline{x}^\alpha)^{\underline{l}} &= -\left(\sum_{r=1}^d \varphi_D(l_r+1,l_r-1) + 2\Gamma^\alpha\right)\;(\underline{x}^\alpha)^{\underline{l}}\,,
\end{aligned}
\tag{4.5}
$$

where $\mathbb{E}^\alpha$ and Γ^α are, respectively, the fractional Euler and Gamma operators

$$
\mathbb{E}^\alpha = \sum_{r=1}^d x_r^\alpha\,\mathcal{D}_r, \qquad\qquad \Gamma^\alpha = -\sum_{r<s} e_r\,e_s\;(x_r^\alpha\,\mathcal{D}_s - \mathcal{D}_r\,x_s^\alpha)\,. \tag{4.6}
$$

Proof. Indeed, we have

$$
\begin{aligned}
\{\mathcal{D},x\}\;(\underline{x}^\alpha)^{\underline{l}} &= -\sum_{r=1}^d (\mathcal{D}_r\,x_r^\alpha - x_r^\alpha\,\mathcal{D}_r)\,(\underline{x}^\alpha)^{\underline{l}} - 2\sum_{r=1}^d x_r^\alpha\,\mathcal{D}_r\;(\underline{x}^\alpha)^{\underline{l}} \\
&= -\left(\sum_{r=1}^d \varphi_D(l_r+1,l_r-1) + 2\mathbb{E}^\alpha\right)\;(\underline{x}^\alpha)^{\underline{l}}\,,
\end{aligned}
$$

and

$$
\begin{aligned}
[\mathcal{D},x]\;(\underline{x}^\alpha)^{\underline{l}} &= -2\sum_{r<s} e_r\,e_s\;(\mathcal{D}_r x_s^\alpha - x_r^\alpha \mathcal{D}_s)\;(\underline{x}^\alpha)^{\underline{l}} - \sum_{r=1}^d (\mathcal{D}_r x_r^\alpha - x_r^\alpha \mathcal{D}_r)\;(\underline{x}^\alpha)^{\underline{l}} \\
&= -\left(\sum_{r=1}^d \varphi_D(l_r+1,l_r-1) + 2\Gamma^\alpha\right)\;(\underline{x}^\alpha)^{\underline{l}}\,.
\end{aligned}
$$

□

From (4.6) we can derive the following remaining Sommen-Weyl relations:

Theorem 4.3. *Given x, $\mathcal{D}$, the fractional Euler operator $\mathbb{E}^\alpha = \sum_{r=1}^d x_r^\alpha \, \mathcal{D}_r$ and $(\underline{x}^\alpha)^{\underline{l}} = \prod_{i=1}^d (x_i^\alpha)^{l_i}$, with $\underline{l} = (l_1, \ldots, l_d)$, and $l = |\underline{l}| = l_1 + \ldots + l_d$ then the Sommen-Weyl relations are given by:*

$$[x, \mathbb{E}^\alpha] \; (\underline{x}^\alpha)^{\underline{l}} = -\sum_{i=1}^d e_i \, \varphi_D \, (l_i + 1, l_i - 1) \; x_i^\alpha \; (\underline{x}^\alpha)^{\underline{l}} \tag{4.7}$$

$$[\mathcal{D}, \mathbb{E}^\alpha] \; (\underline{x}^\alpha)^{\underline{l}} = -\sum_{i=1}^d e_i \, \varphi(l_i, l_i - 1) \, \varphi_D \, (l_i, l_i - 2) \; (x_i^\alpha)^{-1} \; (\underline{x}^\alpha)^{\underline{l}}. \tag{4.8}$$

Proof. By definition, we have

$$\begin{aligned}
[x, \mathbb{E}^\alpha] \; (\underline{x}^\alpha)^{\underline{l}} &= \left(\sum_{i=1}^d x_i^\alpha e_i \sum_{r=1}^d x_r^\alpha \, \mathcal{D}_r - \sum_{r=1}^d x_r^\alpha \, \mathcal{D}_r \sum_{i=1}^d x_i^\alpha e_i \right) \; (\underline{x}^\alpha)^{\underline{l}} \\
&= \sum_{i \neq r}^n (x_i^\alpha e_i x_r^\alpha \, \mathcal{D}_r - x_r^\alpha \, \mathcal{D}_r \, x_i^\alpha e_i) \; (\underline{x}^\alpha)^{\underline{l}} + \sum_{i=r}^n \left(e_i \, (x_r^\alpha)^2 \; \mathcal{D}_i - x_i^\alpha \, \mathcal{D}_i \, x_i^\alpha e_i \right) \; (\underline{x}^\alpha)^{\underline{l}} \\
&= \sum_{i=1}^n \left(e_i \, (x_r^\alpha)^2 \; \mathcal{D}_i - x_i^\alpha \, \mathcal{D}_i \, x_i^\alpha e_i \right) \; (\underline{x}^\alpha)^{\underline{l}} \\
&= -\sum_{i=1}^d e_i \, \varphi_D \, (l_i + 1, l_i - 1) \; x_i^\alpha \; (\underline{x}^\alpha)^{\underline{l}}
\end{aligned}$$

and in the same way,

$$\begin{aligned}
[\mathcal{D}, \mathbb{E}^\alpha] \; (\underline{x}^\alpha)^{\underline{l}} &= \left(\sum_{i=1}^d \mathcal{D}_i^\alpha e_i \sum_{r=1}^d x_r^\alpha \, \mathcal{D}_r - \sum_{r=1}^d x_r^\alpha \, \mathcal{D}_r \sum_{i=1}^d \mathcal{D}_i^\alpha e_i \right) \; (\underline{x}^\alpha)^{\underline{l}} \\
&= \sum_{i=1}^n e_i \, (\mathcal{D}_i^\alpha x_i^\alpha \, \mathcal{D}_i^\alpha - x_i^\alpha \, x_i^\alpha \mathcal{D}_i^\alpha \mathcal{D}_i^\alpha) \; (\underline{x}^\alpha)^{\underline{l}} \\
&= -\sum_{i=1}^d e_i \, \varphi(l_i, l_i - 1) \, \varphi_D \, (l_i, l_i - 2) \; (x_i^\alpha)^{-1} \; (\underline{x}^\alpha)^{\underline{l}}.
\end{aligned}$$

□

In the classic case, the next step would be the construction of a finite dimensional Lie-superalgebra generated by x and $\mathcal{D}$, which in turns lead to an algebra isomorphic to $\mathfrak{osp}(1|2)$. However, this would require the following equalities

$$[x, \mathbb{E}^\alpha] \; (\underline{x}^\alpha)^{\underline{l}} = -x \; (\underline{x}^\alpha)^{\underline{l}}, \qquad [\mathcal{D}, \mathbb{E}^\alpha] \; (\underline{x}^\alpha)^{\underline{l}} = \mathcal{D} \; (\underline{x}^\alpha)^{\underline{l}}.$$

which, in turn, holds if

$$\varphi_D\,(l_i+1,l_i-1)=1, \qquad\qquad \varphi(l_i,l_i-1)\,\varphi_D\,(l_i,l_i-2)=1.$$

This means that the traditional approach sketched cannot be used in our general case. Despite this setback, we can observe that the polynomial space generated by the powers $(\underline{x}^{\alpha})^{\underline{l}}$ allows a decomposition into homogeneous spaces Π_l, where $\Pi_0 = \mathrm{span}\{[1]\}$, and $\Pi_l = \mathrm{span}\left\{x^{\underline{\beta}}\,[1]\right\}$ with $\underline{\beta}=(\beta_1,\ldots,\beta_d)$ and $|\underline{\beta}|=\beta_1+\ldots+\beta_d=l$, and where $[1]$ is the ground-state (usually, given by 1).

4.1.2 Fractional Fischer Decomposition

Based on the previous section we can now define a Fischer inner product for fractional homogeneous polynomials.

Definition 4.1. *A fractional Fischer inner product of two fractional homogeneous polynomials P and Q is given by*

$$\langle P,Q\rangle = \mathrm{Sc}\left[\overline{P\,(\mathcal{D})}\,Q(x)\right]\Big|_{x=0}, \tag{4.9}$$

where $P\,(\mathcal{D})$ is a differential operator obtained by replacing in the polynomial P each variable x_j^{α} by its corresponding fractional derivative $\mathcal{D}_j$.

From (4.9) we immediately get the following lemma.

Lemma 4.4. *For any polynomial P_{l-1} of homogeneity $l-1$ and any polynomial Q_l of homogeneity l it holds*

$$\langle x\,P_{l-1},\,Q_l\rangle = \langle P_{l-1},\mathcal{D}Q_l\rangle\,. \tag{4.10}$$

This fact allows us to prove the following result:

Theorem 4.5. *For each $l\in\mathbb{N}_0$ we have $\Pi_l=\mathcal{M}_l+x\,\Pi_{l-1}$, where Π_l denotes the space of fractional homogeneous polynomials of degree l and $\mathcal{M}_l$ denotes the space of fractional monogenic homogeneous polynomials of degree l. Moreover, the subspaces $\mathcal{M}_k$ and $x\,\Pi_{l-1}$ are orthogonal with respect to the Fischer inner product (4.9).*

Proof. Since $\Pi_l = x\,\Pi_{l-1}+(x\,\Pi_{l-1})^{\perp}$, it suffices to prove that $(x\,\Pi_{l-1})^{\perp}=\mathcal{M}_l$. Assume that $P_l\in\Pi_l$ is in $(x\,\Pi_{l-1})^{\perp}$. Then, we have $\langle x\,P_{l-1},\,P_l\rangle=0$, for all $P_{l-1}\in\Pi_{l-1}$. From (4.10) we get $\langle P_{l-1},\,\mathcal{D}P_l\rangle=0$, for all $P_{l-1}\in\Pi_{l-1}$. Hence, we obtain that $\mathcal{D}P_l=0$, that is $P_l\in\mathcal{M}_l$. This means that $(x\,\Pi_{l-1})^{\perp}\subset\mathcal{M}_l$. Conversely, take $P_l\in\mathcal{M}_l$. Then, for every $P_{l-1}\in\Pi_{l-1}$ we have that

$$\langle x\,P_{l-1},\,P_l\rangle = \langle P_{l-1},\,\mathcal{D}P_l\rangle = \langle P_{l-1},\,0\rangle = 0,$$

from which it follows that $\mathcal{M}_l \subset (x\,\Pi_{l-1})^{\perp}$. Therefore $\mathcal{M}_l = (x\,\Pi_{l-1})^{\perp}$. $\square$

Due to this theorem, we obtain the fractional Fischer decomposition with respect to the fractional Dirac operator $\mathcal{D}$.

Theorem 4.6. *Let P_l be a fractional homogeneous polynomial of degree l. Then*

$$P_l = M_l + x\,M_{l-1} + x^2\,M_{l-2} + \ldots + x^l\,M_0, \tag{4.11}$$

where each M_j denotes a fractional monogenic polynomial of degree j. More specifically,

$$M_0 \in \Pi_0, \qquad \textit{and} \qquad M_l \in \{u \in \Pi_l : \ \mathcal{D}u = 0\}\,.$$

Moreover, the spaces $x^k M_{l-k}$ in (4.11) are orthogonal to each other with respect to the Fischer inner product (4.9) and the above decomposition can be represented in form of an infinite triangle

$$\begin{array}{cccccccc}
\Pi_0 & & \Pi_1 & & \Pi_2 & & \Pi_3 & \\
 & & & & & & & \\
\mathcal{M}_0 & \xleftarrow{\quad\mathcal{D}\quad} & x\,\mathcal{M}_0 & \xleftarrow{\quad\mathcal{D}\quad} & x^2\,\mathcal{M}_0 & \xleftarrow{\quad\mathcal{D}\quad} & x^3\,\mathcal{M}_0 & \ldots \\
 & & \oplus & & \oplus & & \oplus & \\
 & & \mathcal{M}_1 & \xleftarrow{\quad\mathcal{D}\quad} & x\,\mathcal{M}_1 & \xleftarrow{\quad\mathcal{D}\quad} & x^2\,\mathcal{M}_1 & \ldots \\
 & & & & \oplus & & \oplus & \\
 & & & & \mathcal{M}_2 & \xleftarrow{\quad\mathcal{D}\quad} & x\,\mathcal{M}_2 & \ldots \\
 & & & & & & \oplus & \\
 & & & & & & \mathcal{M}_3 & \ldots
\end{array}$$

While in the classic case all the summands in the same row are isomorphic to $\mathrm{Pin}(d)$-modules, and each row is an irreducible module for the Howe dual pair $\mathrm{Pin}(d) \times \mathfrak{osp}(1|2)$ (see [12]), the same cannot be said for the fractional case. The reason for this fact was already stated in the previous subsection where we showed that for a general φ we do not have a finite dimensional superalgebra generated by x and $\mathcal{D}$ isomorphic to $\mathfrak{osp}(1|2)$.

Nevertheless, we have that all the summands in the same row are indeed modules over $\mathbb{C}_d$. Our conjecture is that these modules are invariant under a certain "fractional" Spin group which does not coincide with the classical one. However, such a study is beyond the scope of the present thesis.

The Dirac operator shifts all spaces in the same row to the left, the multiplication by x shifts them to the right, and both of these actions establish isomorphisms between the modules.

From Theorem 4.6 we can derive the following direct extension to the fractional case of the Almansi decomposition:

Theorem 4.7. *For any fractional harmonic polynomial P_l of degree $l \in \mathbb{N}_0$ in a starlike domain Ω in $\mathbb{R}^d$ with respect to 0, i.e.,*

$$\mathcal{D}^2 P_l = 0, \qquad in\ \Omega,$$

there exist uniquely fractional harmonic functions $P_0, P_1, \ldots, P_{l-1}$ such that

$$P_l = P_0 + |x|^2 P_1 + \ldots + |x|^{2(l-1)} P_{l-1}, \quad in\, \Omega.$$

4.1.3 Homogeneous Monogenic Polynomials

In order to obtain an explicit algorithm for the construction of the projection $\pi_{\mathcal{M}}(P_l)$ of a given fractional homogeneous polynomial P_l into the space of fractional homogeneous monogenic polynomials $\mathcal{M}_l$, we start by looking at the dimension of the space of fractional homogeneous monogenic polynomials of degree l. From the Fischer decomposition (4.11) we get

$$\dim(\mathcal{M}_l) = \dim(\Pi_l) - \dim(\Pi_{l-1}),$$

with the dimension of the space of fractional homogeneous polynomials of degree l given by

$$\dim(\Pi_l) = \frac{(l+d-1)!}{l!\,(d-1)!}.$$

This leads to the following theorem:

Theorem 4.8. *The space of fractional homogeneous monogenic polynomials of degree l has dimension*

$$\dim(\mathcal{M}_l) = \frac{(l+d-1)! - l(l+d-2)!}{l!\,(d-1)!} = \frac{(l+d-2)!}{l!\,(d-2)!}.$$

In the classical setting (see [31, 36, 26]) one usually considers the following scheme for the monogenic projection

$$r = a_0 P_l + a_1 x D P_l + a_2 (x)^2 (D)^2 P_l + \ldots + a_l (x)^l (D)^l P_l, \tag{4.12}$$

with $a_j \in \mathbb{R}, j = 0, \ldots, l$, and $a_0 = 1$.

Let us take a look at an example in $d = 3$.

Example 4.2. *If $d = 3$ and $l = 3$ then (4.12) is given by*

$$r = a_0\, P_3 + a_1\, x\mathcal{D}P_3 + a_2\ (x)^2\, (\mathcal{D})^2\, P_3 + a_3\ (x)^3\, (\mathcal{D})^3\, P_3,$$

where $x = e_1 x_1^\alpha + e_2 x_2^\alpha + e_3 x_3^\alpha$ *and* $P_3 = (x_1^\alpha)^2 (x_2^\alpha)^1 (x_3^\alpha)^0$. *In this case, we would like to obtain the real coefficients* a_0, a_1, a_2, a_3 *such that* $\mathcal{D}P_3 = 0$. *Taking into account that*

$$\begin{aligned}
\mathcal{D}\ (\underline{x}^\alpha)^3 &= e_1\,\varphi(2,1)\,x_1^\alpha\,x_2^\alpha + e_2\,\varphi(1,0)\ (x_1^\alpha)^2 \\
(\mathcal{D})^2\ (\underline{x}^\alpha)^3 &= -\varphi(2,0)\,x_2^\alpha \\
(\mathcal{D})^3\ (\underline{x}^\alpha)^3 &= -e_2\,\varphi(2,0)\,\varphi(1,0) \\
\mathcal{D}x\mathcal{D}\ (\underline{x}^\alpha)^3 &= -e_1\,\varphi(2,1)\,x_1^\alpha\,x_2^\alpha\,(2\varphi(2,1) + 2\varphi(1,0)) \\
&\quad - e_2\left[\varphi(1,0)\ (x_1^\alpha)^2\,(\varphi(3,2) + 2\varphi(1,0) + \varphi(2,1)) + \varphi(2,0)\,(x_2^\alpha)^2\right] \\
&\quad + e_3\varphi(2,0)x_2^\alpha\,x_3^\alpha \\
\mathcal{D}\,(x)^2\,(\mathcal{D})^2\ (\underline{x}^\alpha)^3 &= e_1\,\varphi(2,0)\,\varphi(2,1)\,x_1^\alpha x_2^\alpha \\
&\quad + e_2\,\varphi(2,0)\left[\varphi(1,0)\ (x_1^\alpha)^2 + \varphi(3,2)\ (x_2^\alpha)^2 + \varphi(1,0)\ (x_3^\alpha)^2\right] \\
&\quad + e_3\,\varphi(2,0)\,\varphi(2,1)\,x_2^\alpha\,x_3^\alpha \\
\mathcal{D}\,(x)^3\,(\mathcal{D})^3\ (\underline{x}^\alpha)^3 &= e_2\,\varphi(2,0)\,\varphi(1,0)\ \Big[-(x_1^\alpha)^2\,(\varphi(3,2) + 2\varphi(1,0)) - (x_2^\alpha)^2\,(\varphi(3,2) + 2\varphi(1,0)) \\
&\quad - (x_3^\alpha)^2\,(\varphi(3,2) + 2\varphi(1,0))\Big]\,,
\end{aligned}$$

we obtain the equation

$$\begin{aligned}
0 = \mathcal{D}r &= a_0\,\mathcal{D}\ (\underline{x}^\alpha)^3 + a_1\,\mathcal{D}x\mathcal{D}\ (\underline{x}^\alpha)^3 + a_2\mathcal{D}\,(x)^2\,(\mathcal{D})^2\ (\underline{x}^\alpha)^3 + a_3\,\mathcal{D}\,(x)^3\,(\mathcal{D})^3\ (\underline{x}^\alpha)^3 \\
&= (a_0 + a_1\,\mathcal{D}x)\left[e_1\,\varphi(2,1)x_1^\alpha\,x_2^\alpha + e_2\,\varphi(1,0)\ (x_1^\alpha)^2\right] + a_2\,\mathcal{D}\,(x)^2\,[-\varphi(2,0)x_2^\alpha] \\
&+a_3\,\mathcal{D}\,(x)^3\,[-e_2\,\varphi(2,0)\,\varphi(1,0)] = e_1\,\varphi(2,1)\,x_1^\alpha\,x_2^\alpha\,[1 - a_1\,(2\varphi(2,1) + 2\varphi(1,0)) + a_2\,\varphi(2,0)] \\
&+e_2\,\varphi(1,0)\Big\{(x_1^\alpha)^2\,[1 - a_1\,(\varphi(3,2) + 2\varphi(1,0) + \varphi(2,1)) + a_2\varphi(2,0) - a_3\varphi(2,0)\,(\varphi(3,2) + 2\varphi(1,0))] \\
&+\varphi(2,1)\,x_2^\alpha\,[-a_1 + a_2\,\varphi(3,2) - a_3\,(\varphi(3,2) + 2\varphi(1,0))] + \varphi(2,0)\ (x_3^\alpha)^2\,[a_2 - a_3\,(\varphi(3,2) + 2\varphi(1,0))]\Big\} \\
&+e_3\varphi(2,0)\,x_2^\alpha\,x_3^\alpha\,(a_1 + a_2\,\varphi(2,1))\,,
\end{aligned}$$

which leads to the following system of linear equations

$$\begin{cases}
1 - a_1\ (2\varphi(2,1) + 2\varphi(1,0)) + a_2\,\varphi(2,0) = 0 \\
1 - a_1\ (\varphi(3,2) + 2\varphi(1,0) + \varphi(2,1)) + a_2\,\varphi(2,0) - a_3\,\varphi(2,0)\,(\varphi(3,2) + 2\varphi(1,0)) = 0 \\
-a_1 + a_2\,\varphi(3,2) - a_3\ (\varphi(3,2) + 2\varphi(1,0)) = 0 \\
a_2 - a_3\ (\varphi(3,2) + 2\varphi(1,0)) = 0 \\
a_1 + a_2\,\varphi(2,1) = 0
\end{cases},$$

which has no solution if we assume a_0, a_1, a_2, $a_3 \in \mathbb{R}$. *This fact is, of course, due to the lack of the* $\mathfrak{osp}(1|2)$ *property.*

In this example, on can see that we cannot follow the classic approach via monogenic projection (4.12).

Now, by using the Fischer decomposition and the explicit knowledge of the dimensions of the spaces (see Theorem 4.8) we can use a more direct approach to determine the fractional homogeneous monogenic polynomial. In fact, any homogeneous polynomial of degree $l = |\underline{l}|$ can be written as

$$P_l(\underline{x}^\alpha) = \sum_{\underline{l}\in\mathbb{N}_0^d\colon |\underline{l}|=l} (\underline{x}^\alpha)^{\underline{l}}\, a_{\underline{l}}, \qquad a_{\underline{l}} \in \mathbb{R},$$

with $l = |\underline{l}| = l_1 + \ldots + l_d$ denoting the degree of the polynomial. We now check under which conditions we have $\mathcal{D}P_l = 0$, i.e.,

$$\begin{aligned}
0 &= \mathcal{D}\left(\sum_{\underline{l}\in\mathbb{N}_0^d\colon |\underline{l}|=l} (\underline{x}^\alpha)^{\underline{l}}\, a_{\underline{l}}\right) \\
&= \sum_{\underline{l}\in\mathbb{N}_0^d\colon |\underline{l}|=l} \mathcal{D}\,(\underline{x}^\alpha)^{\underline{l}}\, a_{\underline{l}} \\
&= \sum_{\underline{l}\in\mathbb{N}_0^d\colon |\underline{l}|=l} \left(\sum_{j=1}^d e_j\,\mathcal{D}_j\right)\left(\prod_{i=1}^d (x_i^\alpha)^{l_i}\right) a_{\underline{l}} \qquad (4.13)\\
&= \sum_{j=1}^d e_j \left(\sum_{\underline{l}\in\mathbb{N}_0^d\colon |\underline{l}|=l} \varphi(l_j, l_j-1)\,\left(x_j^\alpha\right)^{-1}\,(\underline{x}^\alpha)^{\underline{l}}\, a_{\underline{l}}\right) \\
&= \left[e_1\,\varphi(d,d-1)\,a_{(d,0,\ldots,0)} + e_2\,\varphi(1,0)\,a_{(d-1,1,0,\ldots,0)} + \ldots + e_d\,\varphi(1,0)\,a_{(d-1,0,\ldots,1)}\right]\,(x_1^\alpha)^{d-1} \\
&\quad + \ldots \\
&\quad + \left[e_1\,\varphi(2,1)\,a_{(2,1,\ldots,0)} + e_2\,\varphi(2,1)\,a_{(1,2,1,\ldots,0)} + \ldots + e_d\,\varphi(1,0)\,a_{(1,\ldots,1,0\ldots,0)}\right]\,x_1^\alpha\,x_2^\alpha\ldots x_{d-1}^\alpha.
\end{aligned}$$

The last equality leads to the following theorem

Theorem 4.9. *Equation $\mathcal{D}P_l = 0$ is equivalent to the following linear system*

$$M\,A = 0, \qquad (4.14)$$

for all l, where $A = [a_{(l_1,\ldots,l_d)}]_{dim(\Pi_l)\times 1}$, $0 = [0]_{dim(\Pi_{l-1})\times 1}$ are vectors, and M is the matrix

$$M = \left[M_{(k_1,\ldots,k_d),(l_1,\ldots,l_d)}\right]_{dim(\Pi_{l-1})\times(dim(\Pi_l))},$$

with entrances given by

$$M_{(k_1,\ldots,k_d),(l_1,\ldots,l_d)} = \begin{cases} e_i\,\varphi(l_i,k_i), & k_i = l_i - 1 \wedge k_j = l_j\ \forall i \neq j \\ 0, & \text{other cases} \end{cases}.$$

Let us now indicate a possible ordering for the rows of system (4.14). In order to do that, let us consider the ordered set

$$L = \{\underline{L}^i = (l_1^i, \ldots, l_d^i) : \ |\underline{L}^i| = l = l_1^i + \ldots + l_d^i, \ i = 1, 2 \ldots, dim(\Pi_l)\},$$

where the relation order is given by

$$\underline{L}^i > \underline{L}^{i+1} \qquad \Leftrightarrow \qquad \left(l_1^i, \ldots, l_d^i\right) > \left(l_1^{i+1}, \ldots, l_d^{i+1}\right) \qquad \Leftrightarrow \qquad l_i^i l_2^i \ldots l_d^i > l_1^{i+1} l_2^{i+1} \ldots l_d^{i+1}$$

with

$$l_1^k l_2^k \ldots l_d^k \leftrightarrow l_1^k \times 10^{d-1} + l_2^k \times 10^{d-2} + \ldots + l_d^k \times 10^0.$$

Applying this ordering we get the following corollary.

Corollary 4.10. *The matrix M has the following structure:*

$$M = \begin{pmatrix} M_1 & M_2 \end{pmatrix},$$

where the sub-matrix $M_1 = [m_{ij}^1]_{dim(\Pi_{l-1}) \times dim(\Pi_{l-1})}$ is an upper triangular matrix with entrances given by

$$M_1 = \left(\begin{matrix} e_1\varphi(l, l-1) & e_2\varphi(1,0) & e_3\varphi(1,0) & e_4\varphi(1,0) & e_5\varphi(1,0) \\ 0 & e_1\varphi(l-1, l-2) & 0 & e_2\varphi(2,1) & e_3\varphi(1,0) \\ 0 & 0 & e_1\varphi(l-1, l-2) & 0 & e_2\varphi(1,0) \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \right.$$

$$\left. \begin{matrix} \cdots & e_d\varphi(1,0) & 0 & 0 & \cdots & 0 \\ \cdots & e_{d-2}\varphi(1,0) & e_{d-1}\varphi(1,0) & e_d\varphi(1,0) & \cdots & 0 \\ \cdots & e_{d-3}\varphi(1,0) & e_{d-1}\varphi(1,0) & e_d\varphi(1,0) & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & 0 & 0 & 0 & \cdots & e_1\varphi(1,0) \end{matrix} \right),$$

and the sub-matrix $M_2 = [m^2_{ij}]_{dim(\Pi_{l-1})\times dim(\Pi_l)}$ *has its entrances given by*

$$M_2 = \left(\begin{array}{cccc} 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots \\ e_2\varphi(l,l-1) & e_3\varphi(1,0) & e_4\varphi(1,0) & \cdots \\ 0 & e_2\varphi(l-1,l-2) & e_3\varphi(2,1) & \cdots \\ \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & 0 & 0 \end{array} \right.$$

$$\left. \begin{array}{cccc} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ e_d\varphi(1,0) & 0 & \cdots & 0 \\ e_{d-1}\varphi(1,0) & e_d\varphi(1,0) & \cdots & 0 \\ \ddots & \ddots & \ddots & \vdots \\ 0 & e_2\varphi(1,0) & \cdots & e_d\varphi(l,l-1) \end{array} \right).$$

Since M_1 is upper triangular matrix, for the resolution of system (4.14) we implement the following algorithm to obtain the coefficients.

Theorem 4.11. *Given* $a_{dim(\Pi_{l-1})+1}, \ldots, a_{dim(\Pi_l)}$ *as free parameters and let the entry* (n,n) *of* M_1 *correspond to the index* $[(k_1,k_2,\ldots,k_d),(k_1+1,k_2,\ldots,k_d)]$*, then*

$$a_n = e_1\ (\varphi(k_1+1,k_1))^{-1}\ \left[\sum_{j=2}^{d} e_j\, \varphi(k_j+1,k_j)\, a_{(k_1,\ldots,k_j+1,\ldots,k_d)}\right], \tag{4.15}$$

where

$$a_n \leftrightarrow M_{(n,n)} \leftrightarrow M_{(k_1,l\ldots,k_d),(k_1+1,k_2,\ldots,k_d)}.$$

Example 4.3. *To illustrate the structure of* M *and* A *consider the case* $d=3$, $l=3$, $\alpha=\frac{1}{2}$, *and the Mittag-Leffler function, i.e.* $\varphi(a,b)=\frac{\Gamma(a\alpha+1)}{\Gamma(b\alpha+1)}$ *(see Example 2.2). Taking into account Corollary 4.10, the vector* A *and the matrixes* M_1, M_2 *take the form*

$$\begin{aligned} A^T &= \begin{pmatrix} a_{(3,0,0)} & a_{(2,1,0)} & a_{(2,0,1)} & a_{(1,2,0)} & a_{(1,1,1)} & a_{(1,0,2)} & a_{(0,3,0)} & a_{(0,2,1)} & a_{(0,1,2)} & a_{(0,0,3)} \end{pmatrix} \\ &= \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} \end{pmatrix} \end{aligned}$$

$$M_1 = \begin{pmatrix} e_1\,\varphi(3,2) & e_2\,\varphi(1,0) & e_3\,\varphi(1,0) & 0 & 0 & 0 \\ 0 & e_1\,\varphi(2,1) & 0 & e_2\,\varphi(2,1) & e_3\,\varphi(1,0) & 0 \\ 0 & 0 & e_1\,\varphi(2,1) & 0 & e_2\,\varphi(1,0) & e_3\,\varphi(2,1) \\ 0 & 0 & 0 & e_1\,\varphi(1,0) & 0 & 0 \\ 0 & 0 & 0 & 0 & e_1\,\varphi(1,0) & 0 \\ 0 & 0 & 0 & 0 & 0 & e_1\,\varphi(1,0) \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e_2\,\varphi(3,2) & e_3\,\varphi(1,0) & 0 & 0 \\ 0 & e_2\,\varphi(2,1) & e_3\,\varphi(2,1) & 0 \\ 0 & 0 & e_2\,\varphi(1,0) & e_3\,\varphi(3,2) \end{pmatrix}$$

Now we can see that the columns of the matrix M_2 are associated, respectively, to the last four elements of the matrix A. Hence, if we fix a_7, a_8, a_9, a_{10} we can obtain, via formula (4.15), the remaining elements of the matrix A:

$$\begin{aligned} a_1 &= a_{(3,0,0)} = e_1\ (\varphi(3,2))^{-1} \left[e_2\,\varphi(1,0)\,a_{(2,1,0)} + e_3\,\varphi(1,0)\,a_{(2,0,1)}\right], \\ a_2 &= a_{(2,1,0)} = e_1\ (\varphi(2,1))^{-1} \left[e_2\,\varphi(2,1)\,a_{(1,2,0)} + e_3\,\varphi(1,0)\,a_{(1,1,1)}\right], \\ a_3 &= a_{(2,0,1)} = e_1\ (\varphi(2,1))^{-1} \left[e_2\,\varphi(1,0)\,a_{(1,1,1)} + e_3\,\varphi(2,1)\,a_{(1,0,2)}\right], \\ a_4 &= a_{(1,2,0)} = e_1\ (\varphi(1,0))^{-1} \left[e_2\,\varphi(3,2)\,a_{(0,3,0)} + e_3\,\varphi(1,0)\,a_{(0,2,1)}\right], \\ a_5 &= a_{(1,1,1)} = e_1\ (\varphi(1,0))^{-1} \left[e_2\,\varphi(2,1)\,a_{(0,2,1)} + e_3\,\varphi(2,1)\,a_{(0,1,2)}\right], \\ a_6 &= a_{(1,0,2)} = e_1\ (\varphi(1,0))^{-1} \left[e_2\,\varphi(1,0)\,a_{(0,1,2)} + e_3\,\varphi(3,2)\,a_{(0,0,3)}\right]. \end{aligned}$$

and, therefore, we solve system (4.14). Furthermore, we can use the previous conclusions to obtain the four polynomials which are the basis for the space of fractional homogeneous monogenic polynomials $\mathcal{M}_3$

$$\begin{aligned} V_1^{3,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) &= -e_3\ (x_1^\alpha)^3 - \frac{3}{2}\ (x_1^\alpha)^2\ (x_2^\alpha) + e_3\,\frac{3}{2}\,x_1^\alpha\ (x_2^\alpha)^2 + (x_2^\alpha)^3\,, \\ V_2^{3,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) &= e_2\,\frac{2}{3}\ (x_1^\alpha)^3 - (x_1^\alpha)^2\ x_3^\alpha - e_2\ (x_1^\alpha)\ (x_2^\alpha)^2 + e_3\,\frac{4}{\pi}\,x_1^\alpha\,x_2^\alpha\,x_3^\alpha + (x_2^\alpha)^2\ x_3^\alpha, \\ V_3^{3,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) &= -e_3\,\frac{2}{3}\ (x_1^\alpha)^3 - (x_1^\alpha)^2\ x_2^{2/3} - e_2\,\frac{4}{\pi}\,x_1^\alpha\,x_2^\alpha\,x_3^\alpha + e_3\,x_1^\alpha\ (x_3^\alpha)^2 + x_2^\alpha\ (x_3^\alpha)^2\,, \\ V_4^{3,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) &= e_2\ (x_1^\alpha)^3 - \frac{3}{2}\ (x_1^\alpha)^2\ x_3^\alpha - e_2\,\frac{3}{2}\,x_1^\alpha\ (x_3^\alpha)^2 + (x_3^\alpha)^3\,. \end{aligned}$$

For convenience of the reader we also give the basic monogenic polynomials for $\mathcal{M}_1$ and $\mathcal{M}_2$, *respectively*

$$V_1^{1,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) = -e_3\, x_1^\alpha + x_2^\alpha,$$
$$V_2^{1,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) = -e_2\, x_1^\alpha + x_3^\alpha,$$

$$V_1^{2,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) = -\left(x_1^\alpha\right)^2 + e_3\,\frac{4}{\pi}\,x_1^{2/3}\,x_2^\alpha + \left(x_2^\alpha\right)^2,$$
$$V_2^{2,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) = -e_2\, x_1^\alpha\, x_2^\alpha + e_3\, x_1^\alpha\, x_3^\alpha + x_2^\alpha\, x_3^\alpha,$$
$$V_3^{2,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) = -\left(x_1^\alpha\right)^2 - e_2\,\frac{4}{\pi}\,x_1^\alpha\, x_3^\alpha + \left(x_3^\alpha\right)^2.$$

Remark 4.12. *The above algorithm can be easily implemented. For the convenience of the reader a Matlab program for the DGL operators (see Example 2.2) in the three dimensional case, is available at*

http://sweet.ua.pt/pceres/Webpage/Main_files/Frac_Code.zip

The main program is coef_frac(d,l,a). *This program calculates the coefficients of the monogenic homogeneous polynomials that form the basis of the space of fractional homogeneous monogenic polynomials* $\mathcal{M}_l$, *i.e., solves the system (4.14). The input data of this program consists of the dimension of the space (at this moment is fixed to* d=3*), the degree of homogeneity* l*, and the value of* α *as* a*. For the cases presented in the previous example the program should be called, respectively, in the following form*

coef_frac(3,3,0.5)   coef_frac(3,1,0.5)   coef_frac(3,2,0.5).

The output of the program is given as cell of 4x4 matrices representing quaternions, where the coefficients for each polynomial are given by each column ordered according to Multi-indices given by the function MultiindexIndexgen*. We opted for the matrix representation of quaternions, but the program can be adapted to any Clifford algebra by using the appropriate matrix representation in terms of a* $2^n \times 2^n$ *real or complex matrix (the latter for the case of complexified Clifford algebras).*

4.2 FHMP with respect to Ternary Algebras

The aim of this section is to provide the basic tools for a function theory for the ternary Dirac operator defined via generalized Gelfond-Leontiev differentiation operators. As in the first section, the definition of the corresponding Fischer product is given as well as the Fischer decomposition is presented. We will show that the ternary algebras provide a cubic factorization of the Laplacian and we will present the associated function theory. Moreover, there is an algorithm implemented in MATLAB to compute the coefficients of the monogenic homogeneous polynomials that form the basis of the space of fractional homogeneous monogenic polynomials that arise in the three-dimensional case.

We remark that there are very similar definitions and results to the previous section so that some results have their proofs omitted in this section.

4.2.1 Fractional Fischer Decomposition with respect to Ternary Algebras

As we mentioned before, the standard approach to the establishment of a function theory in higher dimensions is the construction of the analogues to the Euler and Gamma operators and the establishment of the corresponding Sommen-Weyl relations. However, in our case we cannot follow this path directly since our generalized algebra does not have the necessary structure, then we will follow the more classic approach via the Fischer inner product.

We consider the ternary Dirac operator $D = \sum_{j=1}^{d} e_j D_j^\alpha$, where D_j^α represents the fractional derivative associated to the Mittag-Leffler function $E_{\alpha,1}$ (see 2.2) with respect to the j-coordinate ($0 < \alpha < 1$). Therefore, for $x_l \in \mathbb{C}$ we obtain (see (2.22)):

$$D_j^\alpha x_l^0 = 0, \quad D_j^\alpha x_l = \frac{\varphi_0}{\varphi_1}\delta_{j,l} := \varphi(1,0)\delta_{j,l}, \tag{4.16}$$

for all $j, l = 1, \ldots, d$, where

$$\varphi(k,l) = \frac{\Gamma(1+k\alpha)}{\Gamma(1+l\alpha)}, \quad k, l = 0, 1, 2, \ldots, \tag{4.17}$$

which imply that: $D_j^\alpha x_l = \Gamma(1+\alpha)\delta_{j,l}$.

First we can analyze how the differential operators act on the variables x_i:

$$\begin{aligned}
(D_j^\alpha)^k x_j^l &= \varphi(l,l-1)(D_j^\alpha)^{k-1} x_j^{l-1} \\
&= \varphi(l,l-1)\varphi(l-1,l-2)(D_j^\alpha)^{k-2} x_j^{l-2} \\
&\ \vdots \\
&= \varphi(l,l-1)\varphi(l-1,l-2)\cdots\varphi(l-k+1,l-k)x_j^{l-k}. \qquad (4.18)
\end{aligned}$$

Therefore we obtain the next lemma.

Lemma 4.13. *The differential operators $(D_j^\alpha)^k$ act on the variables $x_j = 0$ as*

$$(D_j^\alpha)^k x_j^l \Big|_{x_j=0} = \begin{cases} 0, & \text{if} \quad k \neq l; \\ \varphi(l,l-1)\cdots\varphi(1,0) & \text{if} \quad k = l; \end{cases} \tag{4.19}$$

and we write $\varphi(l,l-1)\cdots\varphi(1,0) = \Gamma(1+l\alpha) := \Phi_l$.

We will begin by analyzing the Fischer decomposition on the right module over $\mathcal{C}\ell_d^{1/3}$ of polynomials, more specifically, on their building blocks, homogeneous polynomials of degree n. In fact, any homogeneous polynomial with coefficients in our algebra can be written as:

$$P_n(\boldsymbol{X}) = \sum_{\boldsymbol{l}\in\mathbb{N}_0^d:\ |\boldsymbol{l}|=n} \boldsymbol{X}^{\boldsymbol{l}}\, a_{\boldsymbol{l}}, \qquad a_{\boldsymbol{l}} \in \mathcal{C}\ell_d^{1/3},$$

with $\boldsymbol{l} \in \mathbb{N}_0^d$, $n = |\boldsymbol{l}| = l_1 + \ldots + l_d$ denoting the degree of the polynomial, $\boldsymbol{X}^{\boldsymbol{l}} = x_1^{l_1} \cdots x_d^{l_d}$ and $a_{\boldsymbol{l}} \in \mathcal{C}\ell_d^{1/3}$ has the form $a_{\boldsymbol{l}} = \sum_{\boldsymbol{\nu}} a_{\boldsymbol{l},\boldsymbol{\nu}} e^{\boldsymbol{\nu}}$.

Using the definition 3.2 for the conjugation in $\mathcal{C}\ell_d^{1/3}$, we can obtain the corresponding definition for the Fischer inner product of two fractional homogeneous polynomials.

Definition 4.2. *The Fischer inner product, in $\mathcal{C}\ell_d^{1/3}$, of two fractional homogeneous polynomials P and Q of degree n is given by*

$$\langle P, Q\rangle = \mathrm{Sc}\left[\overline{P}\,(\boldsymbol{\partial})\ \ Q(\boldsymbol{X})\right]\Bigg|_{x_1=\cdots=x_d=0}, \tag{4.20}$$

where $\overline{P}\,(\boldsymbol{\partial})$ is a differential operator obtained by replacing in the polynomial $\overline{P}$ each variable by its corresponding fractional derivative and Sc represents the scalar part of this product, that is, the coefficient of $e^{(0,\cdots,0)}$.

It is easy to check that (4.20) defines an inner product. Then, for fractional homogeneous polynomials of degree n, $P_n(\boldsymbol{X}) = \sum_{\boldsymbol{l}\in\mathbb{N}_0^d:\ |\boldsymbol{l}|=n} \boldsymbol{X}^{\boldsymbol{l}}\, a_{\boldsymbol{l}}$ and $Q_n(\boldsymbol{X}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d:\ |\boldsymbol{k}|=n} \boldsymbol{X}^{\boldsymbol{k}}\, b_{\boldsymbol{k}}$ with $a_{\boldsymbol{l}}, b_{\boldsymbol{k}} \in \mathcal{C}\ell_d^{1/3}$, we obtain

$$\begin{aligned}
\langle P, Q\rangle &= \sum_{|\boldsymbol{l}|=n}\sum_{|\boldsymbol{k}|=n} \overline{a_{\boldsymbol{l}}} \left((D_d^\alpha)^{l_d}\cdots(D_1^\alpha)^{l_1} x_1^{k_1}\cdots x_d^{k_d}\Big|_{x_1=\cdots=x_d=0}\right) b_{\boldsymbol{k}} \\
&= \sum_{|\boldsymbol{l}|=n} \overline{a_{\boldsymbol{l}}} \left(\prod_{l_j\neq 0} \Phi_{l_j}\right) b_{\boldsymbol{l}} := \sum_{|\boldsymbol{l}|=n} \overline{a_{\boldsymbol{l}}} b_{\boldsymbol{l}} \Phi_{\boldsymbol{l}},
\end{aligned}$$

where $\Phi_{\boldsymbol{l}} = \prod_{j=1,l_j\neq 0}^d \Phi_{l_j}$.

From (4.20) we immediately obtain that, in $\mathcal{C}\ell_d^{1/3}$, for any polynomial P_{n-1} of homogeneity $n-1$ and any polynomial Q_n of homogeneity n the following holds:

$$\langle \boldsymbol{X} P_{n-1},\ Q_n\rangle = \langle P_{n-1}, DQ_n\rangle_{n-1}, \tag{4.21}$$

where $\boldsymbol{X} = x_1 e_1 + \cdots + x_d e_d$, and this fact allows us to prove the corresponding following result:

Theorem 4.14. *For each $n \in \mathbb{N}_0$ we have $\Pi_n = \mathcal{M}_n + X\Pi_{n-1}$, where Π_n denotes the space of fractional homogeneous polynomials of degree n and $\mathcal{M}_n$ denotes the space of fractional* *monogenic homogeneous polynomials of degree n. Moreover, the subspaces $\mathcal{M}_n$ and $X\Pi_{n-1}$ are orthogonal with respect to the Fischer inner product (4.20).*

Proof. We can prove this theorem in the similar way as in the proof for theorem 4.5 given in the last section. □

In consequence, we obtain the fractional Fischer decomposition with respect to the fractional Dirac operator D.

However, in order to obtain further decompositions of the space Π_n we need first to study the commutator relations between the fractional derivatives and variables acting on fractional powers:

$$\begin{aligned}[D_i^\alpha, x_j]\, x_r^l &= (D_i^\alpha x_j - x_j D_i^\alpha)\, x_r^l \\ &= \begin{cases} 0, & \text{if } i \neq j \\ \varphi(1,0)x_r^l, & \text{if } i = j \quad \wedge \quad i \neq r \\ (\varphi(l+1,l) - \varphi(l,l-1))\, x_r^l, & \text{if } i = j = r \end{cases},\end{aligned} \tag{4.22}$$

with $l \in \mathbb{N}$, $i, j, r = 1, \ldots, d$.

Example 4.4. *For the case where $\alpha = 2/3$ we have $\varphi(k,l) = \frac{\Gamma(1+\frac{2k}{3})}{\Gamma(1+\frac{2l}{3})}$. Therefore,*

$$[\mathcal{D}_i, x_j]\, x_r^l = \begin{cases} 0, & \textit{if } i \neq j \\ \Gamma(5/3)x_r^l, & \textit{if } i = j \quad \wedge \quad i \neq r \\ \left(\frac{\Gamma(1+\frac{2(l+1)}{3})}{\Gamma(1+\frac{2l}{3})} - \frac{\Gamma(1+\frac{2l}{3})}{\Gamma(1+\frac{2(l-1)}{3})} \right) x_r^l, & \textit{if } i = j = r \end{cases}.$$

Here we remind the reader that, under certain regularity conditions, the D-G-L derivatives enjoy the semi-group property.

Giving a starlike open domain Ω in $\mathbb{C}^d$ and a (scalar-valued) function $u : \Omega \subset \mathbb{C}^d \to \mathbb{C}$, we have:

$$\Delta^{3\alpha/2} u := D^3 u = (D_1^\alpha)^3 u + \cdots + (D_d^\alpha)^3 u, \quad \text{in } \Omega. \tag{4.23}$$

Analogous to the Euclidean case a $\mathcal{C}\ell_d^{1/3}$-valued function $u : \Omega \subset \mathbb{R}^d \to \mathbb{C}^d$ which are represented as

$$z \mapsto u(z) = \sum_A u_A(z) e_A,$$

with $u_A : \Omega \to \mathbb{C}$ is called *ternary left-monogenic* if it satisfies $Du = 0$ on Ω (resp. *ternary right-monogenic* if it satisfies $uD = 0$ on Ω). As can be seen from the above exposition the most common fractional derivatives arise as special cases in our studies.

Now, we can see that the choice of parameter $\alpha = \frac{2}{3}$ ensures that (4.23) relates to the standard Laplace operator, that is,

$$D^3 u = (\mathcal{D}_1^{2/3})^3 u + \cdots + (\mathcal{D}_d^{2/3})^3 u := \Delta u.$$

Hence, in what follows we shall consider $\alpha = \frac{2}{3}$.

Theorem 4.15. *Let P_n be a fractional homogeneous polynomial of degree n. Then we have:*

$$P_n = M_n + \boldsymbol{X} M_{n-1} + \boldsymbol{X}^2 M_{n-2} + \ldots + \boldsymbol{X}^n M_0, \tag{4.24}$$

where each M_j denotes the fractional monogenic polynomial of degree j. More specifically,

$$M_0 \in \Pi_0, \qquad \textit{and} \qquad M_n \in \{u \in \Pi_n : \quad Du = 0\}.$$

Corollary 4.16. *Let H_n be a fractional homogeneous harmonic polynomial of degree n, i.e., $0 = \Delta H_n := D^3 H_n$. Then H_n has the form:*

$$H_n = M_n + \boldsymbol{X} M_{n-1} + \boldsymbol{X}^2 M_{n-2}. \tag{4.25}$$

The spaces represented in (4.24) are orthogonal to each other with respect to the Fischer inner product (4.20). Moreover, the above decomposition can be represented in form of an infinite triangle:

$$\begin{array}{ccccccccc}
\Pi_0 & & \Pi_1 & & \Pi_2 & & \Pi_3 & & \\
 & & & & & & & & \\
\mathcal{M}_0 & \xleftarrow{\quad D \quad} & \boldsymbol{X}\,\mathcal{M}_0 & \xleftarrow{\quad D \quad} & \boldsymbol{X}^2\,\mathcal{M}_0 & \xleftarrow{\quad D \quad} & \boldsymbol{X}^3\,\mathcal{M}_0 & \ldots \\
 & & \oplus & & \oplus & & \oplus & & \\
 & & \mathcal{M}_1 & \xleftarrow{\quad D \quad} & \boldsymbol{X}\,\mathcal{M}_1 & \xleftarrow{\quad D \quad} & \boldsymbol{X}^2\,\mathcal{M}_1 & \ldots \\
 & & & & \oplus & & \oplus & & \\
 & & & & \mathcal{M}_2 & \xleftarrow{\quad D \quad} & \boldsymbol{X}\,\mathcal{M}_2 & \ldots \\
 & & & & & & \oplus & & \\
 & & & & & & \mathcal{M}_3 & \ldots
\end{array}$$

All the spaces in the above diagrams are right modules over $\mathcal{C}\ell_d^{1/3}$, the Dirac operator shifts all spaces in the same row to the left while the multiplication by X shifts them to the right, and both of these actions establish isomorphisms between the respective modules.

From Theorem 4.15 we can derive the following direct extension to the fractional case of the Almansi decomposition:

Theorem 4.17. *For any fractional polyharmonic polynomial P_n of degree $n \in \mathbb{N}_0$ in a starlike domain Ω in $\mathbb{C}^d$ with respect to 0, i.e.,*

$$D^{3n} P_n = 0, \ \ \textit{in } \Omega,$$

there exist uniquely fractional harmonic functions $P_0, P_1, \ldots, P_{n-1}$ *such that*

$$P_n = P_0 + \boldsymbol{X}^3 P_1 + \ldots + \boldsymbol{X}^{3(n-1)} P_{n-1} \text{ in } \Omega.$$

Corollary 4.18. *Let* H_n *be a fractional homogeneous harmonic polynomial of degree* n, *i.e.,* $0 = \Delta H_n := D^3 H_n$. *Then* H_n *has the form:*

$$H_n = M_n + \boldsymbol{X} M_{n-1} + \boldsymbol{X}^2 M_{n-2}. \tag{4.26}$$

4.2.2 Explicit Construction of FHMP

The aim of this subsection is to give an explicit algorithm for the construction, in the ternary setting, of the projection $\pi_{\mathcal{M}}(P_n)$ of a given fractional homogeneous polynomial P_n into the space of fractional homogeneous monogenic polynomials $\mathcal{M}_n$. In order to reach our goal, we start by looking at the dimension of the space of fractional homogeneous monogenic polynomials of degree n. From the Fischer decomposition (4.24) we obtain:

$$\dim(\mathcal{M}_n) = \dim(\Pi_n) - \dim(\Pi_{n-1}),$$

with the dimension of the space of fractional homogeneous polynomials of degree n given by

$$\dim(\Pi_n) = \frac{(n+d-1)!}{n!(d-1)!} = \binom{n+d-1}{d-1}.$$

This leads to the following theorem:

Theorem 4.19. *The space of fractional homogeneous monogenic polynomials of degree* n *has dimension*

$$\dim(\mathcal{M}_n) = \frac{(n+d-1)!}{n!(d-1)!} - \frac{(n-1+d-1)!}{(n-1)!(d-1)!} = \frac{(n+d-2)!}{n!(d-2)!} = \binom{n+d-2}{d-2}.$$

In the classical setting (see [31, 36, 26]) one usually considers the following scheme for the monogenic projection:

$$r = a_0 P_n + a_1 X D P_n + a_2 X^2 D^2 P_n + \ldots + a_n X^n D^n P_n,$$

with $a_j \in \mathbb{C}, j = 0, \ldots, l$, and $a_0 = 1$. This approach, unfortunately, does not work in our case; this fact is, of course, due to the lack of the $\mathfrak{osp}(1|2)$ property. Using the Fischer decomposition and the explicit knowledge of the dimensions of the spaces (see Theorem 4.19) we can use a more direct approach to determine the fractional homogeneous monogenic polynomial.

As we have seen before, any homogeneous polynomial of degree $n = |\boldsymbol{l}|$ with coefficients in $\mathcal{C}\ell_d^{1/3}$ can be written as

$$P_n(\boldsymbol{X}) = \sum_{\boldsymbol{l}\in\mathbb{N}_0^d:\ |\boldsymbol{l}|=n} \boldsymbol{X}^{\boldsymbol{l}}\ a_{\boldsymbol{l}}, \qquad a_{\boldsymbol{l}} \in \mathcal{C}\ell_d^{1/3},$$

with $n = |\boldsymbol{l}| = l_1 + \ldots + l_d$ denoting the degree of the polynomial. We now check under which conditions we have $DP_n = 0$, i.e.,

$$\begin{aligned}
0 = D\left(\sum_{\boldsymbol{l}\in\mathbb{N}_0^d:\ |\boldsymbol{l}|=n} \boldsymbol{X}^{\boldsymbol{l}}\ a_{\boldsymbol{l}}\right) &= \sum_{\boldsymbol{l}\in\mathbb{N}_0^d:\ |\boldsymbol{l}|=n} \left(D\boldsymbol{X}^{\boldsymbol{l}}\right) a_{\boldsymbol{l}} \\
&= \sum_{\boldsymbol{l}\in\mathbb{N}_0^d:\ |\boldsymbol{l}|=n} \left(\sum_{j=1}^{d} e_j\ \mathcal{D}_j x_1^{l_1}\cdots x_d^{l_d}\right)\ a_{\boldsymbol{l}} \\
&= e_1\varphi(n,n-1)x_1^{n-1}a_{(n,0,\cdots,0)} \\
&+[e_1\varphi(n-1,n-2)x_1^{n-2}x_2 + e_2\varphi(1,0)x_1^{n-1}]a_{(n-1,1,0,\cdots,0)} \\
&+\cdots+[e_1\varphi(n-1,n-2)x_1^{n-2}x_d + e_d\varphi(1,0)x_1^{n-1}]a_{(n-1,0,\cdots,0,1)} \\
&+[e_1\varphi(n-2,n-3)x_1^{n-3}x_2^2 + e_2\varphi(2,1)x_1^{n-2}x_2]a_{(n-2,2,0,\cdots,0)} \\
&+[e_1\varphi(n-2,n-3)x_1^{n-3}x_2x_3 + e_2\varphi(1,0)x_1^{n-2}x_3 + e_3\varphi(1,0)x_1^{n-2}x_2]a_{(n-2,1,1,0\cdots,0)} \\
&+\cdots+[e_1\varphi(n-2,n-3)x_1^{n-3}x_2^2 + e_d\varphi(2,1)x_1^{n-2}x_d]a_{(n-2,0,\cdots,0,2)} \\
&+\cdots+e_d\varphi(n,n-1)x_d^{n-1}a_{(0,\cdots,0,n)}.
\end{aligned} \tag{4.27}$$

The last equality leads to the following theorem:

Theorem 4.20. *Equation (4.27) is equivalent to the following linear system:*

$$MA = 0, \tag{4.28}$$

where $A = [a_{(l_1,\ldots,l_d)}]_{dim(\Pi_n)\times 1}$, $0 = [0]_{dim(\Pi_{n-1})\times 1}$ *are vectors, and* M *is the matrix*

$$M = \left[M_{(k_1,\ldots,k_d),(l_1,\ldots,l_d)}\right]_{dim(\Pi_{n-1})\times dim(\Pi_n)},$$

with entrances given by

$$M_{(k_1,\ldots,k_d),(l_1,\ldots,l_d)} = \begin{cases} e_i\ \varphi(l_i,k_i), & k_i = l_i - 1 \wedge k_j = l_j \quad \forall i \neq j \\ 0, & \textit{others cases} \end{cases}.$$

Let us now indicate a possible ordering for the rows of system (4.28). In order to proceed, let us consider the following ordered set:

$$L = \{\underline{L}^i = (l_1^i,\ldots,l_d^i):\quad |\underline{L}^i| = n = l_1^i + \ldots + l_d^i,\quad i = 1,2\ldots,dim(\Pi_n)\},$$

where the relation order is given by

$$\underline{L}^i > \underline{L}^{i+1} \quad \Leftrightarrow \quad \left(l_1^i, \ldots, l_d^i\right) > \left(l_1^{i+1}, \ldots, l_d^{i+1}\right) \quad \Leftrightarrow \quad l_i^i l_2^i \ldots l_d^i > l_1^{i+1} l_2^{i+1} \ldots l_d^{i+1}$$

with

$$l_1^k l_2^k \ldots l_d^k \leftrightarrow l_1^k \times 10^{d-1} + l_2^k \times 10^{d-2} + \ldots + l_d^k \times 10^0.$$

Applying this ordering we get the following corollary.

Corollary 4.21. *The matrix M has the following structure:*

$$M = \begin{pmatrix} M_1 & M_2 \end{pmatrix},$$

where the sub-matrix $M_1 = [m_{ij}^1]_{dim(\Pi_{n-1}) \times dim(\Pi_{n-1})}$ is an upper triangular matrix with entrances given by:

$$M_1 = \left(\begin{array}{ccccc} e_1\varphi(n,n-1) & e_2\varphi(1,0) & e_3\varphi(1,0) & e_4\varphi(1,0) & e_5\varphi(1,0) \\ 0 & e_1\varphi(n-1,n-2) & 0 & e_2\varphi(2,1) & e_3\varphi(1,0) \\ 0 & 0 & e_1\varphi(n-1,n-2) & 0 & e_2\varphi(1,0) \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & 0 \end{array} \right.$$

$$\left. \begin{array}{cccccc} \cdots & e_d\varphi(1,0) & 0 & 0 & \cdots & 0 \\ \cdots & e_{d-2}\varphi(1,0) & e_{d-1}\varphi(1,0) & e_d\varphi(1,0) & \cdots & 0 \\ \cdots & e_{d-3}\varphi(1,0) & e_{d-1}\varphi(1,0) & e_d\varphi(1,0) & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & 0 & 0 & 0 & \cdots & e_1\varphi(1,0) \end{array} \right),$$

and the sub-matrix $M_2 = [m_{ij}^2]_{dim(\Pi_{n-1}) \times dim(\Pi_n)}$ has its entrances given by:

$$M_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ e_2\varphi(n,n-1) & e_3\varphi(1,0) & e_4\varphi(1,0) & \cdots & e_d\varphi(1,0) & 0 & \cdots & 0 \\ 0 & e_2\varphi(n-1,n-2) & e_3\varphi(2,1) & \cdots & e_{d-1}\varphi(1,0) & e_d\varphi(1,0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & e_2\varphi(1,0) & \cdots & e_d\varphi(n,n-1) \end{pmatrix}.$$

For the resolution of system (4.28) we implement the following algorithm to obtain the coefficients. As a first step, we re-ordered the coefficients $a_{(n,0,\cdots,0)}, a_{(n-1,1,0,\cdots,0)}, \cdots, a_{(0,\cdots,0,n)}$ as $a_1, a_2, \cdots, a_{dim(\Pi_n)}$. Second, we use the fact that M_1 is an upper triangular matrix. Let the entry (i,i) of M_1 correspond to the index $[(k_1,k_2,\ldots,k_d),(k_1+1,k_2,\ldots,k_d)]$, then

$$a_i = -e_1^2 \ \left(\varphi(k_1+1,k_1)\right)^{-1} \left[\sum_{j=2}^{d} e_j \ \varphi(k_j+1,k_j) \ a_{(k_1,\ldots,k_j+1,\ldots,k_d)}\right], \tag{4.29}$$

where

$$a_i \leftrightarrow M_{(i,i)} \leftrightarrow M_{(k_1,k_2,\ldots,k_d),(k_1+1,k_2,\ldots,k_d)}.$$

For the implementation of the algorithm we use the following matrix representation:

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}.$$

$$E_1^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad E_2^2 = \begin{pmatrix} 0 & 0 & 1 \\ \omega^2 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}, \qquad E_3^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix},$$

$$E_1E_2 = \begin{pmatrix} 0 & 0 & \omega^2 \\ 1 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}, \qquad E_2E_3 = \begin{pmatrix} 0 & \omega^2 & 0 \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{pmatrix}.$$

This representation determines a sub-algebra of $\mathcal{C}\ell_3^{1/3}$, yielding the extra condition $E_1E_3 = E_2$.

Example 4.5. *To illustrate the structure of M and A, consider the case of $d=3$ and the Mittag-Leffer function $E_{\frac{2}{3},1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+2k/3)}$. We recall that $\varphi(a,b) = \frac{\Gamma(1+\frac{2a}{3})}{\Gamma(1+\frac{2b}{3})}$ (see Example 2.2). Taking into account Corollary 4.21, the vector A and the matrices M_1, M_2 take the form*

$$\begin{aligned} A^T &= \begin{pmatrix} a_{(3,0,0)} & a_{(2,1,0)} & a_{(2,0,1)} & a_{(1,2,0)} & a_{(1,1,1)} & a_{(1,0,2)} & a_{(0,3,0)} & a_{(0,2,1)} & a_{(0,1,2)} & a_{(0,0,3)} \end{pmatrix} \\ &= \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} \end{pmatrix} \end{aligned}$$

$$M_1 = \begin{pmatrix} E_1\ \varphi(3,2) & E_2\ \varphi(1,0) & E_3\ \varphi(1,0) & 0 & 0 & 0 \\ 0 & E_1\ \varphi(2,1) & 0 & E_2\ \varphi(2,1) & E_3\ \varphi(1,0) & 0 \\ 0 & 0 & E_1\ \varphi(2,1) & 0 & E_2\ \varphi(1,0) & E_3\ \varphi(2,1) \\ 0 & 0 & 0 & E_1\ \varphi(1,0) & 0 & 0 \\ 0 & 0 & 0 & 0 & E_1\ \varphi(1,0) & 0 \\ 0 & 0 & 0 & 0 & 0 & E_1\ \varphi(1,0) \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E_2\ \varphi(3,2) & E_3\ \varphi(1,0) & 0 & 0 \\ 0 & E_2\ \varphi(2,1) & E_3\ \varphi(2,1) & 0 \\ 0 & 0 & E_2\ \varphi(1,0) & E_3\ \varphi(3,2) \end{pmatrix}.$$

Now we can see that the columns of the matrix M_2 are associated, respectively, to the last four elements of the matrix A. Therefore, if we fix a_6, a_7, a_8, a_8 we can obtain, via formula (4.29), the remaining elements of the matrix A as follows:

$$\begin{aligned} a_1 &= -E_1^2\ (\varphi(3,2))^{-1} \left[E_2\ \varphi(1,0)\ a_{(2,1,0)} + E_3\ \varphi(1,0)\ a_{(2,0,1)}\right], \\ a_2 &= -E_1^2\ (\varphi(2,1))^{-1} \left[E_2\ \varphi(2,1)\ a_{(1,2,0)} + E_3\ \varphi(1,0)\ a_{(1,1,1)}\right], \\ a_3 &= -E_1^2\ (\varphi(2,1))^{-1} \left[E_2\ \varphi(1,0)\ a_{(1,1,1)} + E_3\ \varphi(2,1)\ a_{(1,0,2)}\right], \\ a_4 &= -E_1^2\ (\varphi(1,0))^{-1} \left[E_2\ \varphi(3,2)\ a_{(0,3,0)} + E_3\ \varphi(1,0)\ a_{(0,2,1)}\right], \\ a_5 &= -E_1^2\ (\varphi(1,0))^{-1} \left[E_2\ \varphi(2,1)\ a_{(0,2,1)} + E_3\ \varphi(2,1)\ a_{(0,1,2)}\right], \\ a_6 &= -E_1^2\ (\varphi(1,0))^{-1} \left[E_2\ \varphi(1,0)\ a_{(0,1,2)} + E_3\ \varphi(3,2)\ a_{(0,0,3)}\right], \end{aligned}$$

which concludes the solution to the system (4.28). Furthermore, we can use the previous conclusions to obtain the four polynomials which are the basis for the space of fractional *homogeneous monogenic polynomials* $\mathcal{M}_3$

$$\begin{aligned}
V_1^{3,\frac{2}{3}}\left(x^{2/3}\right) &= -x_1^3 + \frac{27\sqrt{3}}{8\pi}\ E_3^2\ x_1^2\ x_2 - \frac{27\sqrt{3}}{8\pi}\ E_3\ x_1\ x_2^2 + x_2^3, \\
V_2^{3,\frac{2}{3}}\left(x^{2/3}\right) &= -E_2^2\ x_1^2\ x_2 + E_3^2\ x_1^2\ x_3 - E_1E_2\ x_1\ x_2^2 \\
&\quad -\frac{\Gamma\left(\frac{7}{3}\right)}{\Gamma^2\left(\frac{5}{3}\right)}\ E_3\ x_1\ x_2\ x_3 + x_2^2\ x_3, \\
V_3^{3,\frac{2}{3}}\left(x^{2/3}\right) &= \omega\ E_2E_3\ x_1^2\ x_2 - E_2^2\ x_1^2\ x_3 \\
&\quad -\frac{\Gamma\left(\frac{7}{3}\right)}{\Gamma^2\left(\frac{5}{3}\right)}\ E_1^2E_3\ x_1\ x_2\ x_3 - E_3\ x_1\ x_3^2 + x_2\ x_3^2, \\
V_4^{3,\frac{2}{3}}\left(x^{2/3}\right) &= -x_1^3 + \frac{27\sqrt{3}}{8\pi}\ \omega\ E_2E_3\ x_1^2\ x_3 \\
&\quad -\frac{27\sqrt{3}}{8\pi}\ E_1E_2\ x_1\ x_3^2 + x_3^3.
\end{aligned}$$

For the convenience of the reader we will also write the basic monogenic polynomials for $\mathcal{M}_1$ *and* $\mathcal{M}_2$*, respectively, as follows:*

$$\begin{aligned}
V_1^{1,\frac{2}{3}}\left(x^{2/3}\right) &= -E_3\ x_1 + x_2, \\
V_2^{1,\frac{2}{3}}\left(x^{2/3}\right) &= -E_1E_2\ x_1 + x_3.
\end{aligned}$$

$$\begin{aligned}
V_1^{2,\frac{2}{3}}\left(x^{2/3}\right) &= \omega^{-2}\ E_3^2\ x_1^2 - \frac{\Gamma\left(\frac{7}{3}\right)}{\Gamma^2\left(\frac{5}{3}\right)}\ E_3\ x_1\ x_2 + x_2^2, \\
V_2^{2,\frac{2}{3}}\left(x^{2/3}\right) &= -\frac{\Gamma^2\left(\frac{5}{3}\right)}{\Gamma\left(\frac{7}{3}\right)}\ E_2^2\ x_1^2 - E_1E_2\ x_1\ x_2 - E_3\ x_1\ x_3 \\
&\quad +x_2\ x_3, \\
V_3^{2,\frac{2}{3}}\left(x^{2/3}\right) &= \omega^{-2}\ E_2E_3\ x_1^2 \\
&\quad -\frac{\Gamma\left(\frac{7}{3}\right)}{\Gamma^2\left(\frac{5}{3}\right)}\ E_1E_2\ x_1\ x_3 + x_3^2.
\end{aligned}$$

Remark 4.22. *The above algorithm can be easily implemented. For the convenience of the reader a Matlab program for the D-G-L operators (see Example 2.2) is available at*

http://sweet.ua.pt/pceres/Webpage/Main_files/Frac_Code_ternary.zip

The main program is coef_frac(l)*. This program calculates the coefficients of the monogenic homogeneous polynomials that form the basis of the space of fractional homogeneous monogenic polynomials* $\mathcal{M}_l$*, i.e., solves the system (4.28). The output is given as cells of* 3×3

representing the linear combination of the elements

$$\left\{I_3,\ E_1,\ E_2,\ E_3,\ E_1^2,\ E_2^2,\ E_3^2,\ E_1E_2,\ E_2E_3\right\}, \tag{4.30}$$

where the coefficients for each polynomial are given by each column ordered according to Multi-indices given by the function MultiindexIndexgen*. The input data of this program consists of the degree of homogeneity* l*.*

The auxilliar program coef_frac_final_form(A{r,c}) *reads each cell of the output of the main program and presents the coefficients involved in the linear combination indicated previously. The input of this program is each cell of the output of the main program.*

For the case presented in the previous example first we should call the main program in the form coef_frac(3) *to generate all the coefficients. After that, in order to obtain the coefficients of V_1 (similarly for V_2, V_3 and V_4) we make*

coef_frac_final_form(A{1,1}), coef_frac_final_form(A{2,1}),
coef_frac_final_form(A{3,1}),
coef_frac_final_form(A{4,1}), coef_frac_final_form(A{5,1}),
coef_frac_final_form(A{6,1}),
coef_frac_final_form(A{7,1}), coef_frac_final_form(A{8,1}),
coef_frac_final_form(A{9,1}),
coef_frac_final_form(A{10,1}).

Chapter 5

Fractional Cauchy-Kovalevskaya Theorem and Reproducing Kernel Hilbert Modules

In this section, we turn our attention to the Cauchy-Kovalevskaya(CK) extension theorem in the generalized fractional setting by using the Gelfond-Leontiev derivative operators with respect to the classic Clifford algebras and the ternary algebras. We will also present, in both cases, the reproducing kernel Hilbert modules that arise from the monogenic formal powers constructed via the CK theorem.

As we will see, these monogenic formal powers constructed via the CK theorem allow us to follow ideas from [7, 8] to construct reproducing kernel Hilbert spaces of monogenic functions, in particular the Drury-Arveson space and the de Branges-Rovnyak spaces associated to Schur multipliers. Furthermore, with the basic monogenic powers and Fueter series we will study the Gleason's problem in this context and its link with Leibenson's shift operators.

5.1 The Case of the Clifford Algebras

Given $\{e_1, \cdots, e_n\}$ be the standard basis of $\mathbb{R}^d$ we recall that the associated Clifford algebra $\mathbb{R}_{0,d}$ is the free algebra generated by $\mathbb{R}^d$ subject to the multiplication rules $e_ie_j+e_je_i = -2\delta_{i,j}$, where $i, j = 1, \ldots d$.

Let D_φ be the fractional Cauchy-Riemann operator

$$D_\varphi = \partial_0^\varphi + \sum_{j=1}^{d} e_j\partial_j^\varphi = \partial_0^\varphi + e_1\partial_1^\varphi + \cdots + e_d\partial_d^\varphi, \tag{5.1}$$

where ∂_j^α $(j = 0, 1, \ldots d)$ represents the Gelfond-Leontiev generalized derivative with respect to the $j-$coordinate and the entire function φ with order $\rho > 0$ and type $\sigma > 0$, that is, such that

$$\lim_{k\to\infty} k^{\frac{1}{\rho}} \sqrt[k]{|\varphi_k|} = (\sigma e \rho)^{\frac{1}{\rho}}$$

.

5.1.1 Fractional Cauchy-Kovalevskaya Extension

The aim of this section is to give a new extension of the Cauchy-Kovalevskaya theorem that arises in the case of the Gelfond-Leontiev operator of generalized differentiation with respect to an entire function φ which can be used to find the basis for the space of fractional homogeneous monogenic polynomials.

In the following theorem, we give the fractional Cauchy-Kovalevskaya theorem.

Theorem 5.1 (Fractional Cauchy-Kovalevskaya extension)**.** *Let $P_{\boldsymbol{\nu}}$ be a homogeneous product given by $P_{\boldsymbol{\nu}}(x_1,\cdots,x_d) := x_1^{\nu_1}\cdots x_d^{\nu_d}$, $\boldsymbol{\nu}=(\nu_1,\cdots,\nu_d)\in\mathbb{N}_0^d$ and let φ be an entire function with order $\rho>0$ and type $\sigma\neq 0$. The Cauchy-Kovalevskaya extension $CK_\varphi[P_{\boldsymbol{\nu}}]$ of $P_{\boldsymbol{\nu}}$ has the form*

$$CK_\varphi[P_{\boldsymbol{\nu}}](x_0,x_1,\cdots,x_d) := \left[\varphi\left(-x_0\underline{D}_\varphi\right)\right] x_1^{\nu_1}\cdots x_d^{\nu_d}. \tag{5.2}$$

Moreover, $CK_\varphi[P_{\boldsymbol{\nu}}]$ is a monogenic polynomial homogeneous of degree $|\boldsymbol{\nu}| := \nu_1+\cdots+\nu_d$ and $\underline{D}_\varphi := \sum_{j=1}^d e_j\partial_j^\varphi$ with ∂_j^φ is the Gelfond-Leontiev operator of generalized differentiation with respect to the function φ.

Proof. Using $D_\varphi = \partial_0^\varphi + \underline{D}_\varphi$ we have

$$\begin{aligned}
D_\varphi CK_\varphi[P_{\boldsymbol{\nu}}](x_0,x_1,\cdots,x_d) &= D_\varphi\left[\varphi\left(-x_0\underline{D}_\varphi\right)\right] x_1^{\nu_1}\cdots x_d^{\nu_d}\\
&= (\partial_0^\varphi + \underline{D}_\varphi)\sum_{k=0}^\infty \varphi_k(-1)^k(x_0)^k(\underline{D}_\varphi)^k(x_1^{\nu_1}\cdots x_d^{\nu_d})
\end{aligned}$$

$$= \sum_{k=1}^\infty \varphi_k(-1)^k\frac{\varphi_{k-1}}{\varphi_k}(x_0)^{k-1}(\underline{D}_\varphi)^k(x_1^{\nu_1}\cdots x_d^{\nu_d}) + \sum_{k=0}^\infty \varphi_k(-1)^k(x_0)^k(\underline{D}_\varphi)^{k+1}(x_1^{\nu_1}\cdots x_d^{\nu_d})$$

$$= \sum_{k=0}^\infty \varphi_k(-1)^{k+1}(x_0)^k(\underline{D}_\varphi)^{k+1}(x_1^{\nu_1}\cdots x_d^{\nu_d}) + \sum_{k=0}^\infty \varphi_k(-1)^k(x_0)^k(\underline{D}_\varphi)^{k+1}(x_1^{\nu_1}\cdots x_d^{\nu_d})$$

$$= \sum_{k=0}^\infty[-\varphi_k+\varphi_k](-1)^k(x_0)^k(\underline{D}_\varphi)^{k+1}(x_1^{\nu_1}\cdots x_d^{\nu_d}) = 0.$$

□

In the next corollary, we obtain the case of theorem (5.1) for the Mittag-Leffler function $E_{\alpha,1}(\lambda)$.

Corollary 5.2. *If $\varphi(\lambda)=E_{\alpha,1}(\lambda)$ is the Mittag-Leffler function then the Cauchy-Kovalevskaya extension of a homogeneous product $P_{\boldsymbol{\nu}}(x_1,\cdots,x_d) := x_1^{\nu_1}\cdots x_d^{\nu_d}$, $\boldsymbol{\nu}=(\nu_1,\cdots,\nu_d)\in\mathbb{N}_0^d$ is* *given by*

$$CK_\alpha[P_{\boldsymbol{\nu}}](x_0, x_1, \cdots, x_d) := \left[E_{\alpha,1}\left(-x_0\underline{D}_\varphi\right)\right] x_1^{\nu_1}\cdots x_d^{\nu_d}. \tag{5.3}$$

For more details about this previous corollary we refer to [8].

Example 5.1. *In the Example (2.3), we can see that if* $\varphi(\lambda) = E_{1,1}(\lambda) = \exp(\lambda)$ *then* $\underline{D}_\varphi = \sum_{j=1}^d e_j\partial_j^\varphi = \sum_{j=1}^d e_j\frac{d}{dx_j} = D_x$*. Therefore,*

$$CK_\alpha[P_{\boldsymbol{\nu}}](x_0, x_1, \cdots, x_d) = \left[E_{1,1}\left(-x_0\underline{D}_\varphi\right)\right] x_1^{\nu_1}\cdots x_d^{\nu_d} = \exp(-x_0D_x),$$

so that we have the Cauchy-Kovalevskaya extension in the classical Clifford analysis (see 1.4).

The importance of the Theorem (5.1) is due to the fact that it establishes an isomorphism between the space of fractional monogenics generated by $(x_0, x_1, \ldots, x_d)$ with coefficients in the fractional Clifford algebra and the space of fractional homogeneous polynomials in the variables $(x_1, \ldots, x_d)$ with fractional Clifford-valued coefficients.

Definition 5.1. *Let* $\boldsymbol{\nu} = (\nu_1, \cdots, \nu_d) \in \mathbb{N}_0^d$ *with* $|\boldsymbol{\nu}| = \nu_1 + \cdots + \nu_d$*. Then the fractional monogenic powers*

$$\zeta^{\boldsymbol{\nu}}(x) = CK_\varphi[P_{\boldsymbol{\nu}}](x_0, x_1, \ldots, x_d) := \sum_{j=0}^{|\boldsymbol{\nu}|}(-1)^j(x_0)^j\varphi_j(\underline{D}_\varphi)^j(x_1^{\nu_1}\cdots x_d^{\nu_d}) \tag{5.4}$$

are called the fractional Fueter polynomials of degree $|\boldsymbol{\nu}|$*.*

Example 5.2. *We have* $\zeta^0(x) = \varphi_0$*, while for* $\boldsymbol{\nu} = (0, \cdots, \nu_j, \cdots, 0)$ *with* $\nu_j = 1$*, that is,* $\boldsymbol{\nu} = e_j$

$$\begin{aligned}\zeta^{e_j}(x) &= \sum_{j=0}^{1}(-1)^j(x_0)^j\varphi_j(\underline{D}_\varphi)^j(x_1^0\cdots x_j^1\cdots x_d^0)\\ &= \varphi_0x_j - x_0\varphi_1\frac{\varphi_0}{\varphi_1}e_j = \varphi_0(x_j - x_0e_j) \quad j = 1, \ldots, d.\end{aligned}$$

Example 5.3. *In order to find the basis for the space of fractional homogeneous monogenic polynomials* $\mathcal{M}_l$ *(see theorem 4.15), we consider the case* $d = 3$*,* $l = 3$ *and the Mittag-Leffler function* $\varphi(\lambda) = E_{\alpha,1}(\lambda)$ *with* $\varphi_k = \frac{1}{\Gamma(k\alpha+1)}$ *(see corollary 5.2). In this case, we obtain the two polynomials which are the basis for the space* $\mathcal{M}_1$*:*

$$\begin{aligned}V_1^{1,\alpha}\left(\underline{x}^\alpha\right) &= -e_2\, x_1^\alpha + x_2^\alpha,\\ V_2^{1,\alpha}\left(\underline{x}^\alpha\right) &= -e_3\, x_1^\alpha + x_3^\alpha;\end{aligned}$$

the three polynomials which are the basis for the space $\mathcal{M}_2$:

$$\begin{aligned}
V_1^{2,\alpha}(\underline{x}^\alpha) &= -\frac{\varphi_2}{\varphi_1}(x_1^\alpha)^2 - e_2\,\frac{\varphi_1^2}{\varphi_2}\,x_1^\alpha\,x_2^\alpha + (x_2^\alpha)^2\,,\\
V_2^{2,\alpha}(\underline{x}^\alpha) &= -e_3\,x_1^\alpha\,x_2^\alpha - e_2\,x_1^\alpha\,x_3^\alpha + x_2^\alpha\,x_3^\alpha,\\
V_3^{2,\alpha}(\underline{x}^\alpha) &= -\frac{\varphi_2}{\varphi_1}(x_1^\alpha)^2 - e_3\,\frac{\varphi_1^2}{\varphi_2}\,x_1^\alpha\,x_3^\alpha + (x_3^\alpha)^2
\end{aligned}$$

and the four polynomials which are the basis for the space $\mathcal{M}_3$:

$$\begin{aligned}
V_1^{3,\alpha}(\underline{x}^\alpha) &= e_2\,(x_1^\alpha)^3 - e_2\frac{\varphi_2\varphi_1}{\varphi_3}\,(x_1^\alpha)^2\,(x_2^\alpha) - e_2\,\frac{\varphi_2\varphi_1}{\varphi_3}\,x_1^\alpha\,(x_2^\alpha)^2 + (x_2^\alpha)^3\,,\\
V_2^{3,\alpha}(\underline{x}^\alpha) &= e_3\,\frac{\varphi_3}{\varphi_1\varphi_2}\,(x_1^\alpha)^3 - e_2\,(x_1^\alpha)^2\,x_3^\alpha - e_3\,(x_1^\alpha)\,(x_2^\alpha)^2 - e_2\,\frac{\varphi_1^2}{\varphi_2}\,x_1^\alpha\,x_2^\alpha\,x_3^\alpha + (x_2^\alpha)^2\,x_3^\alpha,\\
V_3^{3,\alpha}(\underline{x}^\alpha) &= e_2\,\frac{\varphi_3}{\varphi_1\varphi_2}\,(x_1^\alpha)^3 - e_3\,(x_1^\alpha)^2\,x_2^\alpha - e_3\,\frac{\varphi_1^2}{\varphi_2}\,x_1^\alpha\,x_2^\alpha\,x_3^\alpha - e_2\,x_1^\alpha\,(x_3^\alpha)^2 + x_2^\alpha\,(x_3^\alpha)^2\,,\\
V_4^{3,\alpha}(\underline{x}^\alpha) &= e_3\,(x_1^\alpha)^3 - e_3\,\frac{\varphi_2\varphi_1}{\varphi_3}\,(x_1^\alpha)^2\,x_3^\alpha - e_3\,\frac{\varphi_2\varphi_1}{\varphi_3}\,x_1^\alpha\,(x_3^\alpha)^2 + (x_3^\alpha)^3\,.
\end{aligned}$$

In this above example, we can remark that if $\alpha = \dfrac{1}{2}$ we have (unless of some constant) the same elements of the basis in the Example (4.3).

As the Cauchy-Kovalevskaya extension establishes an isomorphism between the space of fractional homogeneous polynomials and the space of fractional monogenic polynomials with coefficients in the fractional Clifford algebra, then it allows us to determine the basis for the space of fractional homogeneous monogenic polynomials.

Theorem 5.3. *If $P_{\boldsymbol{\nu}}(x_1,\cdots,x_d) := x_1^{\nu_1}\cdots x_d^{\nu_d},\ \boldsymbol{\nu} = (\nu_1,\cdots,\nu_d) \in \mathbb{N}_0^d$ is a homogeneous product then the set $\{\zeta^{\boldsymbol{\nu}}|\zeta^{\boldsymbol{\nu}}(x) = CK_\varphi[P_{\boldsymbol{\nu}}]\}$ is a basis for the space of fractional homogeneous monogenic polynomials $\mathcal{M}_l$.*

It is known that given f a $\mathbb{R}_d$-valued real analytic function near the origin then f admits a decomposition

$$f(x) = \sum_{|\boldsymbol{\nu}|=0}^{\infty} P_{\boldsymbol{\nu}}(x_1,\cdots,x_d), \quad \boldsymbol{\nu} = (\nu_1,\cdots,\nu_d)$$

where $P_{\boldsymbol{\nu}}(x_1,\cdots,x_d)$ is a homogeneous $\mathbb{R}_d$-valued polynomial of degree $|\boldsymbol{\nu}| := \nu_1 + \cdots + \nu_d$ in $x_1^{\nu_1}\cdots x_d^{\nu_d}$.

Since homogeneous polynomials are building blocks for real analytic functions in $\mathbb{R}_d$, it is of fundamental importance to know their Cauchy-Kovalevskaya extension and then we can get the Cauchy-Kovalevskaya extension for $\mathbb{R}_d$-valued real analytic function.

5.1.2 Reproducing Kernel Hilbert Modules

To start, we now define the right Clifford module of fractional power series by using the fractional monogenic powers given by the Cauchy-Kovalevskaya extension.

Definition 5.2. *Let $\zeta^{\boldsymbol{\nu}}$ be the fractional monogenic powers defined in (5.4). Then the right Clifford module $\mathcal{M}$ of the space of fractional monogenic powers is defined by*

$$f(x) = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \zeta^{\boldsymbol{\nu}}(x) f_{\boldsymbol{\nu}} = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} CK_{\varphi}[P_{\boldsymbol{\nu}}](x_0, x_1, \cdots, x_d) f_{\boldsymbol{\nu}}, \tag{5.5}$$

where $\sum_{\boldsymbol{\nu}} |f_{\boldsymbol{\nu}}|^2 < \infty$.

The series in (5.5) is called fractional Fueter series since for $\varphi(\lambda) = E_{1,1}(\lambda)$ (see Example (5.1)) it corresponds to the classical Fueter series. Moreover, we may observe that by the construction of f we can easily get that $Df = 0$ for all $f \in \mathcal{M}$.

We define $(\partial_{\varphi})^{\boldsymbol{\mu}}$ as

$$(\partial_{\varphi})^{\boldsymbol{\mu}} := (\partial_1^{\varphi})^{\mu_1} \cdots (\partial_d^{\varphi})^{\mu_d},$$

where ∂_i^{φ} $(i = 1, \cdots, d)$ is the GL operator of generalized differentiation with respect to x_i. Then $(\partial_{\varphi})^{\boldsymbol{\mu}}$ acts on elements $x^{\boldsymbol{\nu}} = x_1^{\nu_1} \cdots x_d^{\nu_d}$ as

$$\begin{aligned} (\partial_{\varphi})^{\boldsymbol{\mu}}(x_1^{\nu_1} \cdots x_d^{\nu_d}) &= (\partial_1^{\varphi})^{\mu_1} \cdots (\partial_d^{\varphi})^{\mu_d} x_1^{\nu_1} \cdots x_d^{\nu_d} \\ &= \Pi_{j=1}^d \varphi(\nu_j, \nu_j - \mu_j) x_1^{\nu_1 - \mu_1} \cdots x_d^{\nu_d - \mu_d}, \end{aligned} \tag{5.6}$$

where $\varphi(\nu_j, \nu_j - \mu_j) = \dfrac{\varphi_{\nu_j - \mu_j}}{\varphi_{\nu_j}}$ and $\nu_j \geq \mu_j$. In the case of $\nu_j < \mu_j$ we have $\varphi(\nu_j, \nu_j - \mu_j) = 0$.

Lemma 5.4. *Given a function f defined in (5.5) we have*

$$(\partial_{\varphi})^{\boldsymbol{\mu}} f(x) = \sum_{l=0}^{\infty} \sum_{|\boldsymbol{\nu}|=l} \varphi(\boldsymbol{\nu}, \boldsymbol{\nu} - \boldsymbol{\mu}) \zeta^{\boldsymbol{\nu} - \boldsymbol{\mu}}(x) f_{\boldsymbol{\nu}},$$

where $\varphi(\boldsymbol{\nu}, \boldsymbol{\nu} - \boldsymbol{\mu}) = \Pi_{j=1}^d \varphi(\nu_j, \nu_j - \mu_j) = \Pi_{j=1}^d \dfrac{\varphi_{\nu_j - \mu_j}}{\varphi_{\nu_j}}$. *Moreover, if there exists j $(j = 1, \cdots, d)$ such that $\nu_j < \mu_j$ then* $\varphi(\boldsymbol{\nu}, \boldsymbol{\nu} - \boldsymbol{\mu}) = 0$.

Proof. In fact, we have that

$$
\begin{aligned}
(\partial_\varphi)^{\boldsymbol{\mu}} f(x) &= \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} (\partial_\varphi)^{\boldsymbol{\mu}} \zeta^{\boldsymbol{\nu}}(x) f_{\boldsymbol{\nu}} \\
&= \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \sum_{l=0}^{|\boldsymbol{\nu}|} \varphi_l (-x_0 \underline{D}_\varphi)^l (\partial_1^\varphi)^{\mu_1} \cdots (\partial_d^\varphi)^{\mu_d} (x_1^{\nu_1} \cdots x_d^{\nu_d}) f_{\boldsymbol{\nu}} \\
&= \sum_{k=1}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \Pi_{j=1}^d \varphi(\nu_j, \nu_j - \mu_j) \sum_{l=0}^{|\boldsymbol{\nu}-\boldsymbol{\mu}|} \varphi_l (-x_0 \underline{D}_\varphi)^l (x_1^{\nu_1-\mu_1} \cdots x_d^{\nu_d-\mu_d}) f_{\boldsymbol{\nu}} \\
&= \sum_{k=1}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \varphi(\boldsymbol{\nu}, \boldsymbol{\nu}-\boldsymbol{\mu}) \zeta^{\boldsymbol{\nu}-\boldsymbol{\mu}}(x) f_{\boldsymbol{\nu}}
\end{aligned}
$$

where $\varphi(\boldsymbol{\nu}, \boldsymbol{\nu}-\boldsymbol{\mu}) = \Pi_{j=1}^d \varphi(\nu_j, \nu_j - \mu_j)$. □

However, for an arbitrary $\boldsymbol{\nu}$ such that $|\boldsymbol{\nu}| = k_0$ we get

$$
(\partial_\varphi)^{\boldsymbol{\nu}} f(x) = \varphi(\boldsymbol{\nu}, 0)\varphi_0 f_{\boldsymbol{\nu}} + \sum_{k=1}^{\infty} \sum_{|\boldsymbol{\mu}|=k,\ \boldsymbol{\mu}\neq\boldsymbol{\nu}} \varphi(\boldsymbol{\mu}, \boldsymbol{\mu}-\boldsymbol{\nu}) \zeta^{\boldsymbol{\mu}-\boldsymbol{\nu}}(x) f_{\boldsymbol{\mu}}
$$

since $\zeta^0(x) = \varphi_0$.

We now consider the Clifford-valued coefficients $f_{\boldsymbol{\nu}}$ defined as

$$
f_{\boldsymbol{\nu}} := \frac{1}{\varphi(\boldsymbol{\nu}, 0)\varphi_0} \left. (\partial_\varphi)^{\boldsymbol{\nu}} f(x) \right|_{x=0}. \tag{5.7}
$$

Using (5.7) we define a product between monogenic power series.

Definition 5.3. *Let $f, g \in \mathcal{M}$ with series expansions given by*

$$
f(x) = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \zeta^{\boldsymbol{\nu}}(x) f_{\boldsymbol{\nu}}, \qquad g(x) = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\mu}|=k} \zeta^{\boldsymbol{\mu}}(x) g_{\boldsymbol{\mu}}.
$$

We define a Cauchy product between f and g as

$$
(f \otimes g)(x) := \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \zeta^{\boldsymbol{\nu}}(x) \left(\sum_{0 \le |\boldsymbol{\mu}| \le |\boldsymbol{\nu}|} f_{\boldsymbol{\mu}} g_{\boldsymbol{\nu}-\boldsymbol{\mu}} \right). \tag{5.8}
$$

For the particular case of $f(x) = \zeta^{e_i}(x)$ and $g(x) = \zeta^{e_j}(x)$, $i \neq j$, the Cauchy product between f and g is given by

$$\begin{aligned}
(\zeta^{e_i}\otimes\zeta^{e_j})(x) &= \sum_{k=0}^{2}\sum_{|\boldsymbol{\nu}|=k}\zeta^{\boldsymbol{\nu}}(x)\left(\sum_{0\le|\boldsymbol{\mu}|\le|\boldsymbol{\nu}|} f_{\boldsymbol{\mu}}g_{\boldsymbol{\nu}-\boldsymbol{\mu}}\right)\\
&= \sum_{|\boldsymbol{\nu}|=0}\zeta^{\boldsymbol{\nu}}(x)\left(\sum_{0\le|\boldsymbol{\mu}|\le 0} f_{\boldsymbol{\mu}}g_{\boldsymbol{\nu}-\boldsymbol{\mu}}\right)+\sum_{|\boldsymbol{\nu}|=1}\zeta^{\boldsymbol{\nu}}(x)\left(\sum_{0\le|\boldsymbol{\mu}|\le 1} f_{\boldsymbol{\mu}}g_{\boldsymbol{\nu}-\boldsymbol{\mu}}\right)\\
&\quad+\sum_{|\boldsymbol{\nu}|=2}\zeta^{\boldsymbol{\nu}}(x)\left(\sum_{0\le|\boldsymbol{\mu}|\le 2} f_{\boldsymbol{\mu}}g_{\boldsymbol{\nu}-\boldsymbol{\mu}}\right)\\
&= 0+[\zeta^{e_i}(x)(0+0)+\zeta^{e_j}(x)(0+0)]+\zeta^{e_i+e_j}(x)(0+1+0)\\
&= \zeta^{e_i+e_j}(x).
\end{aligned}$$

We can also obtain that $(\zeta^{e_i}\otimes\zeta^{e_j})(x)=(\zeta^{e_j}\otimes\zeta^{e_i})(x)$ since

$$(\zeta^{e_i}\otimes\zeta^{e_j})(x)=\zeta^{e_i+e_j}(x)=\zeta^{e_j+e_i}(x)=(\zeta^{e_j}\otimes\zeta^{e_i})(x)$$

and consequently, $(\zeta^{e_j}\otimes\zeta^{\boldsymbol{\nu}-e_j})(x)=\zeta^{\boldsymbol{\nu}}(x)$.

We remark that the product (5.8) coincides with the pointwise one in some cases of these functions, for instance, if $g(x)=\text{const}$ then $f\otimes g=fg$.

5.1.3 Reproducing Kernel Right-Hilbert Module

Having introduced the definition of the right Clifford module of fractional power series as well as defined a product between monogenic power series, we arrive now at the question of reproducing kernel right-Hilbert module (RKHM) arising from the monogenic formal powers defined in (5.4).

We start with the following definition.

Definition 5.4. *Given a sequence $c=(c_{\boldsymbol{\nu}}),\ c_{\boldsymbol{\nu}}\ge 0$, for all $\boldsymbol{\nu}\in\mathbb{N}^d$, we define its support as*

$$\operatorname{supp}(c):=\{\boldsymbol{\nu}\in\mathbb{N}^d : c_{\boldsymbol{\nu}}\neq 0\}. \tag{5.9}$$

We now consider the domain

$$\Omega_c:=\left\{x=x_0+x_1e_1+\cdots+x_de_d\in\mathbb{R}^{n+1}:\sum_{k=0}^{\infty}\sum_{\boldsymbol{\nu}\in\operatorname{supp}(c),|\boldsymbol{\nu}|=k} c_{\boldsymbol{\nu}}|\zeta^{\boldsymbol{\nu}}(x)|^2<\infty\right\}. \tag{5.10}$$

and then we define the kernel

$$k_c(x,y)=\sum_{k=0}^{\infty}\sum_{\boldsymbol{\nu}\in\operatorname{supp}(c),|\boldsymbol{\nu}|=k} c_{\boldsymbol{\nu}}\zeta^{\boldsymbol{\nu}}(x)\overline{\zeta^{\boldsymbol{\nu}}(y)}$$

associated to the domain Ω_c.

Let $\mathcal{H}(c)$ be the associated reproducing kernel right-Hilbert module. It is said $f(x) = \sum_{\boldsymbol{\nu}\in \operatorname{supp}(c)} \zeta^{\boldsymbol{\nu}}(x) f_{\boldsymbol{\nu}}$ belongs to $\mathcal{H}(c)$ if it satisfies

$$\|f\|_c^2 := \sum_{\boldsymbol{\nu}\in \operatorname{supp}(c)} \frac{|f_{\boldsymbol{\nu}}|^2}{c_{\boldsymbol{\nu}}} < \infty.$$

Moreover, we have the reproducing formula

$$f(x) = \int_{\Omega_c} k_c(x,y) f(y) dy$$

which preserves right-linearity with respect to Clifford-valued constants.

From (5.6), by taking $\boldsymbol{\mu} = e_j = (0, \cdots, 1, \cdots, 0)$ for $j = 1, \cdots, d$, it follows that

$$\begin{aligned}
(\partial_j^{\varphi})^{e_j} \zeta^{\boldsymbol{\nu}}(x) &= (\partial_j^{\varphi})^{e_j} \sum_{l=0}^{|\boldsymbol{\nu}|} \varphi_l(-x_0 \underline{D}_{\varphi})^l (x_1^{\nu_1} \cdots x_d^{\nu_d}) \\
&= \sum_{l=0}^{|\boldsymbol{\nu}|} \varphi_l(-x_0 \underline{D}_{\varphi})^l (\partial_j^{\varphi})^{e_j} (x_1^{\nu_1} \cdots x_d^{\nu_d}) \\
&= \varphi(\nu_j, \nu_j - 1) \sum_{l=0}^{|\boldsymbol{\nu}-e_j|} \varphi_l(-x_0 \underline{D}_{\varphi})^l (x_1^{\nu_1} \cdots x_d^{\nu_j - 1} \cdots x_d^{\nu_d}) \\
&= \varphi(\nu_j, \nu_j - 1) \zeta^{\boldsymbol{\nu} - e_j}(x)
\end{aligned}$$

where $\boldsymbol{\nu} - e_j = (\boldsymbol{\nu}_1, \cdots, \boldsymbol{\nu}_j - 1, \cdots, \boldsymbol{\nu}_d)$ and $\boldsymbol{\nu}_j \geq 1$.

Let

$$|\varphi(\boldsymbol{\nu}, \boldsymbol{\nu} - 1)| = \sum_{j=1}^{d} \varphi(\nu_j, \nu_j - 1),$$

where $\nu_j \geq 1$ and for the case $\nu_j = 0$, we get $\varphi(\nu_j, \nu_j - 1) = 0$ since $(\partial_j^{\varphi})^{e_j} \zeta^{\boldsymbol{\nu}}(x) = 0$. We can now introduce the Leibenson's shift operator $R_j : \mathcal{H}(c) \to \mathcal{H}(c)$ given as

$$R_j f(x) = \frac{(\partial_j^{\varphi})^{e_j}}{|\varphi(\boldsymbol{\nu}, \boldsymbol{\nu} - 1)|} f(x) = \sum_{k=1}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \zeta^{\boldsymbol{\nu} - e_j}(x) \frac{\varphi(\nu_j, \nu_j - 1)}{|\varphi(\boldsymbol{\nu}, \boldsymbol{\nu} - 1)|} f_{\boldsymbol{\nu}}. \tag{5.11}$$

This operator is bounded in $\mathcal{H}(c)$ iff the set $\operatorname{supp}(c)$ is lower inclusive and it holds that

$$\|R_j f\|_c^2 = \sum_{\boldsymbol{\nu}\in \operatorname{supp}(c)} \frac{\left| \frac{\varphi(\nu_j, \nu_j - 1)}{|\varphi(\boldsymbol{\nu}, \boldsymbol{\nu}-1)|} f_{\boldsymbol{\nu}} \right|^2}{c_{\boldsymbol{\nu} - e_j}} < \infty$$

or equivalently,

$$\sup_j \left| \frac{\varphi(\nu_j, \nu_j - 1)}{|\varphi(\boldsymbol{\nu}, \boldsymbol{\nu} - 1)|} \right|^2 \frac{c_{\boldsymbol{\nu}}}{c_{\boldsymbol{\nu} - e_j}} \quad < \quad \infty, \tag{5.12}$$

since $\sum_{\boldsymbol{\nu} \in \operatorname{supp}(c)} \dfrac{|f_{\boldsymbol{\nu}}|^2}{c_{\boldsymbol{\nu}}} < \infty$.

5.1.4 A Gleason's Type Problem

The aim of this section is to consider the Gleason problem with the pointwise product being replaced by the Cauchy product (5.8).

The version of Gleason's problem is given as following.

Definition 5.5 (Gleason problem)**.** *Let $\mathcal{M}$ be a set of functions which are fractional monogenic in a neighbourhood of the origin and let $\boldsymbol{\nu} = e_j = (0, \cdots, 1, \cdots, 0)$. The Gleason problem for $f \in \mathcal{M}$ is to find functions $p_1, \ldots, p_d \in \mathcal{M}$ such that*

$$f(x) - f(0) = \sum_{j=1}^{d} (\zeta^{e_j} \otimes p_j)(x). \tag{5.13}$$

It turns out that the Leibenson's shift operators (5.11) provide a solution for this new Gleason's problem as we can see in the following theorem.

Theorem 5.5. *Gleason's problem is solvable in the reproducing kernel right-Hilbert module $\mathcal{H}(c)$ where the weights c satisfy (5.12) and our Leibenson's shift operator's provide the only commutative solution of the problem.*

Proof. Indeed,

$$\sum_{j=1}^{d} \zeta^{e_j} \otimes R_j f(x) = \sum_{j=1}^{d} \zeta^{e_j} \otimes \sum_{k=1}^{\infty} \sum_{|\nu|=k} (\zeta^{\boldsymbol{\nu} - e_j})(x) \frac{\varphi(\nu_j, \nu_j - 1)}{|\varphi(\boldsymbol{\nu}, \boldsymbol{\nu} - 1)|} f_{\boldsymbol{\nu}}$$

$$\sum_{k=1}^{\infty} \sum_{|\nu|=k} \sum_{j=1}^{d} (\zeta^{e_j} \otimes \zeta^{\boldsymbol{\nu} - e_j})(x) \frac{\varphi(\nu_j, \nu_j - 1)}{|\varphi(\boldsymbol{\nu}, \boldsymbol{\nu} - 1)|} f_{\boldsymbol{\nu}} = \sum_{k=1}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \zeta^{\boldsymbol{\nu}}(x) \sum_{j=1}^{d} \frac{\varphi(\nu_j, \nu_j - 1)}{|\varphi(\boldsymbol{\nu}, \boldsymbol{\nu} - 1)|} f_{\boldsymbol{\nu}}$$

$$= \sum_{k=1}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \zeta^{\boldsymbol{\nu}}(x) f_{\boldsymbol{\nu}} = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \zeta^{\boldsymbol{\nu}}(x) f_{\boldsymbol{\nu}} - \zeta^0(x) f_0 = f(x) - \zeta^0(0) f_0$$

$$= f(x) - f(0),$$

where it is used that $\sum_{j=1}^{d} \frac{\varphi(\nu_j, \nu_j - 1)}{|\varphi(\boldsymbol{\nu}, \boldsymbol{\nu} - 1)|} = 1$ and $(\zeta^{e_j} \otimes \zeta^{\boldsymbol{\nu} - e_j})(x) = \zeta^{\boldsymbol{\nu}}(x)$. We can also see that these operators commute to each other:

$$R_jR_lf(x) = \sum_{k=2}^{\infty}\sum_{|\boldsymbol{\nu}|=k}\zeta^{\boldsymbol{\nu}-e_j-e_l}(x)\frac{\varphi(\nu_j,\nu_j-1)}{|\varphi(\boldsymbol{\nu},\boldsymbol{\nu}-1)|}\frac{\varphi(\nu_l,\nu_l-1)}{|\varphi(\boldsymbol{\nu},\boldsymbol{\nu}-1)|}f_{\boldsymbol{\nu}} = R_lR_jf(x).$$

Let $(T_j)_{j=1}^d$ and $(\widetilde{T}_j)_{j=1}^d$ be two families of commuting bounded operators on $\mathcal{H}(c)$ which solve Gleason's problem. By applying the formula (5.13) for T_j we get

$$\begin{aligned}
f(x) &= f(0) + \sum_{j=1}^{d}(\zeta^{e_j}\otimes T_jf)(x) && (5.14)\\
&= f(0) + \sum_{j=1}^{d}\left(\zeta^{e_j}(x)\otimes\left(T_jf(0)+\sum_{l=1}^{d}\zeta^{e_l}\otimes T_lT_jf(x)\right)\right) && (5.15)\\
&= f(0) + \sum_{j=1}^{d}\zeta^{e_j}(x)T_jf(0) + \sum_{j,l=1}^{d}\left(\zeta^{e_j}\otimes\zeta^{e_l}\otimes T_lT_jf\right)(x).
\end{aligned}$$

Again applying the formula (5.13) for T_lT_jf in (5.14), we get

$$\begin{aligned}
f(x) &= f(0) + \sum_{j=1}^{d}\zeta^{e_j}(x)T_jf(0) + \sum_{j,l=1}^{d}\left(\zeta^{e_j}\otimes\zeta^{e_l}\otimes T_lT_jf\right)(x)\\
&= f(0) + \sum_{j=1}^{d}\zeta^{e_j}(x)T_jf(0) + \sum_{j,l=1}^{d}\left(\zeta^{e_j}\otimes\zeta^{e_l}\right)(x)T_lT_jf(0)\\
&\quad + \sum_{j,l,m=1}^{d}\left(\zeta^{e_j}\otimes\zeta^{e_l}\otimes\zeta^{e_m}\otimes T_lT_jf\right)(x)\\
&= f(0) + \sum_{j=1}^{d}\zeta^{e_j}(x)T_jf(0) + \sum_{j<l=1}^{d}2\zeta^{e_i+e_j}(x)T_lT_jf(0)\\
&\quad + \sum_{j=1}^{d}(\zeta^{e_j})^2(x)T_lT_jf(0) + \sum_{j,l,m=1}^{d}\left(\zeta^{e_j}\otimes\zeta^{e_l}\otimes\zeta^{e_m}\otimes T_lT_jf\right)(x)
\end{aligned}$$

Iterating this process, one obtains an expansion of f in terms of products of ζ^{e_j} and then the power series expansion of f leads to

$$f(x) = \sum_{k=0}^{\infty}\sum_{|\boldsymbol{\nu}|=k}\frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!}\zeta^{\boldsymbol{\nu}}(x)T^{\boldsymbol{\nu}}f(0),$$

where $\boldsymbol{\nu}! = \prod_{k=1}^{d}\boldsymbol{\nu}_k!$ and $T^{\boldsymbol{\nu}} = T_1^{\nu_1}\cdots T_d^{\nu_d}$. As we have $f_{\boldsymbol{\nu}} = \dfrac{1}{\varphi(\boldsymbol{\nu},0)\varphi_0}\,(\partial_\varphi)^{\boldsymbol{\nu}}f(x)|_{x=0}$, in particular we obtain

$$T^{\boldsymbol{\nu}} f(0) = \frac{\boldsymbol{\nu}!}{|\boldsymbol{\nu}|!} \frac{1}{\varphi(\boldsymbol{\nu}, 0)\varphi_0} \left. (\partial_\varphi)^{\boldsymbol{\nu}} f(x) \right|_{x=0} = \frac{\boldsymbol{\nu}!}{|\boldsymbol{\nu}|!} f_{\boldsymbol{\nu}}.$$

In the same way for $\widetilde{T}_j$, we get

$$f(x) = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!} \zeta^{\boldsymbol{\nu}}(x) \widetilde{T}^{\boldsymbol{\nu}} f(0),$$

where

$$\widetilde{T}^{\boldsymbol{\nu}} f(0) = \frac{\boldsymbol{\nu}!}{|\boldsymbol{\nu}|!} f_{\boldsymbol{\nu}}.$$

We now have that $T^{\boldsymbol{\nu}} f(0) = \widetilde{T}^{\boldsymbol{\nu}} f(0)$ and thus,

$$(T_j f)(x) = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!} \zeta^{\boldsymbol{\nu}}(x) T^{\boldsymbol{\nu}+e_j} f(0) = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!} \zeta^{\boldsymbol{\nu}}(x) \widetilde{T}^{\boldsymbol{\nu}+e_j} f(0) = (\widetilde{T}_j f)(x),$$

that is, there is only one family of commutative operators that are solution of Gleason's problem, which correspond to Leibenson's shift operators.

□

5.1.5 Fractional Monogenic Operator-Valued Functions

To start this section we recall the definition of a (left or right) Clifford Hilbert module.

Definition 5.6. *X is a left Hilbert $\mathbb{C}_d$-module if X is a left $\mathbb{C}_d$-module and X is also a real Hilbert space such that for any $a \in \mathbb{C}_d$ and $x \in X$:*

$$\|ax\|_X \leq C\ |a| \|x\|_X, \tag{5.16}$$

for some $C > 0$. Similarly one can define a right Banach $\mathbb{C}_d$-module.

Given a right-Clifford Hilbert module $\mathcal{H}$ with the sesqui-linear form

$$\langle f, g \rangle = \int_{\mathbb{R}^{d+1}} f(x)\overline{g(x)} dx.$$

Then its dual space $\mathcal{H}^*$ is also a right-linear Clifford Hilbert module.

Definition 5.7. *Let $\Omega \subset \mathbb{R}^{d+1}$ be a domain containing the origin. A mapping $f : \Omega \mapsto \mathcal{H}^*$ is called an $\mathcal{H}^*$-valued (left-) monogenic function in Ω if for all $h \in \mathcal{H}$ we have $f(\cdot)h$ being a (left-) monogenic function in Ω.*

As in the classical case (see [7]) it follows the following theorem.

Theorem 5.6. *Let f be an $\mathcal{H}^*$-valued monogenic function in a ball $B(0,R)$ centered at the origin with radius R. Then f has a representation via*

$$f(x) = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \zeta^{\boldsymbol{\nu}}(x) f_{\boldsymbol{\nu}}$$

where $f_{\boldsymbol{\nu}} \in \mathcal{H}^*$ are linear functionals over $\mathcal{H}$. Hereby, the series converges normally in $B(0,R)$.

Proof. First of all, the set of functionals $f(x) : |x| \leq R'$ is uniformly bounded for each $R' < R$, that is,

$$\sup_{|x| \leq R'} |f(x)| < \infty.$$

Choosing an arbitrary $h \in \mathcal{H}$ and taking

$$f(x)h = \sum_{k=0}^{\infty} P_k(x,h),$$

which is the expansion of $f(x)h$ into a series of fractional homogeneous polynomials of x, we have that for the terms $P_k(x,h)$ it follows $|P_k(x,h)| \leq C\|h\| \sup_{|x|\leq R'} \|f(x)\|$.

We now have that $P_k(x,h)$ is linear in h, that is, $P_k(x,h) = P_k(x)h$ with $P_k(x) \in \mathcal{H}^*$ and the series $\sum_k P_k(x)$ converges normally in $B(0,R)$ with respect to the operator norm. As $P_k(x)h$ is fractional monogenic in x we can write it as

$$P_k(x)h = \sum_{|\boldsymbol{\nu}|=k} \zeta^{\boldsymbol{\nu}}(x) f_{\boldsymbol{\nu}}(h)$$

with $f_{\boldsymbol{\nu}}(h)$ being linear and bounded in h.

Therefore, this theorem is proved as we desired.

□

In the next corollary, which is a straightforward consequence of the previous theorem, we have a particular representation of the reproducing kernel $k(x,y)$ where is used the dual of the RKHS associated to $k = k(\cdot,\cdot)$.

Corollary 5.7. *A reproducing kernel $k(x,y)$ can be represented as*

$$k(x,y) = g(x)g(y)^*$$

where $g(x)$ is an $\mathcal{H}(k)^*$-valued monogenic function and $\mathcal{H}(k)^*$ denotes the dual of the RKHS associated to $k = k(\cdot,\cdot)$.

Proof. As $k(x,y)$ is a reproducing kernel we have

$$k(x,y) = \langle k(\cdot,y), k(\cdot,x)\rangle.$$

Introducing the operator $g(x)$ such that $g(x)^*1 = k(\cdot,x)$ is the operator of point evaluation we get immediately this corollary. □

5.1.6 Reproducing Kernel Spaces

In this section, we introduce a reproducing kernel Hilbert space of fractional monogenic functions in a domain Ω_c in $\mathbb{R}^{d+1}$.

Let remind that the fractional complex power of a real variable $x^\alpha \in \mathbb{C}$ is understood as

$$x^\alpha := \begin{cases} \exp(\alpha \;\ln|x|); & x > 0 \\ 0; & x = 0 \\ \exp(\alpha \;\ln|x| + i\alpha\pi); & x < 0 \end{cases},$$

with $0 < \alpha < 1$.

If $x = x_0^\alpha + x_1^\alpha e_1 + \cdots + x_d^\alpha e_d$ and $y = y_0^\alpha + y_1^\alpha e_1 + \cdots + y_d^\alpha e_d$ then we have the definition of fractional Drury-Arveson space (or module).

Definition 5.8. *We define the fractional Drury-Arveson space (or module) $\mathcal{A}$ as the RKHS with the reproducing kernel $k_c(x,y)$ given by*

$$k_{\mathcal{A}}(x,y) = \sum_{k=0}^{\infty} \sum_{\boldsymbol{\nu}\in \mathrm{supp}(c), |\boldsymbol{\nu}|=k} c_{\boldsymbol{\nu}} \zeta^{\boldsymbol{\nu}}(x)\overline{\zeta^{\boldsymbol{\nu}}(y)},$$

where the coefficients $c = c_{\boldsymbol{\nu}}$ satisfy (5.12).

Theorem 5.8. *Let C be the operator of evaluation at the origin, that is*

$$Cf := \frac{1}{\varphi(\boldsymbol{\nu},0)\varphi_0}\; ((\partial^\alpha)^{\boldsymbol{\nu}} f(x))|_{x=0}$$

and let M_{ζ_j} be the multiplication operator $M_{\zeta_j} f = f \otimes \zeta_j$. Then we have

$$(I - \sum_{j=1}^{d} M_{\zeta_j} M_{\zeta_j}^*) = C^*C \tag{5.17}$$

if and only if f belongs to the fractional Drury-Arveson space $\mathcal{A}$.

Proof. Applying on both sides of the operator identity (5.17) to the kernel k_c, we get

$$
\begin{aligned}
&(I-\sum_{j=1}^{d} M_{\zeta_j}M_{\zeta_j}^{*})\sum_{k=0}^{\infty}\sum_{\boldsymbol{\nu}\in\operatorname{supp}(c),|\boldsymbol{\nu}|=k} c_{\boldsymbol{\nu}}\zeta^{\boldsymbol{\nu}}(x)\overline{\zeta^{\boldsymbol{\nu}}(y)} \qquad (5.18)\\
&=\sum_{k=0}^{\infty}\sum_{\boldsymbol{\nu}\in\operatorname{supp}(c),|\boldsymbol{\nu}|=k} c_{\boldsymbol{\nu}}\left[\zeta^{\boldsymbol{\nu}}(x)\overline{\zeta^{\boldsymbol{\nu}}(y)}-\sum_{j=1}^{d}\left(\zeta^{\boldsymbol{\nu}}\otimes\zeta^{e_j}\right)(x)\overline{\left(\zeta^{\boldsymbol{\nu}}\otimes\zeta^{e_j}\right)(y)}\right]\\
&=\sum_{k=0}^{\infty}\sum_{\boldsymbol{\nu}\in\operatorname{supp}(c),|\boldsymbol{\nu}|=k} c_{\boldsymbol{\nu}}\left[\zeta^{\boldsymbol{\nu}}(x)\overline{\zeta^{\boldsymbol{\nu}}(y)}-\sum_{j=1}^{d}\zeta^{\boldsymbol{\nu}+e_j}(x)\overline{\zeta^{\boldsymbol{\nu}+e_j}(y)}\right]\\
&=c_0\zeta^{0}(x)\overline{\zeta^{0}(y)}+\sum_{k=1}^{\infty}\sum_{\boldsymbol{\nu}\in\operatorname{supp}(c),|\boldsymbol{\nu}|=k}\left[c_{\boldsymbol{\nu}}-\sum_{j=1}^{d}c_{\boldsymbol{\nu}-e_j}\right]\zeta^{\boldsymbol{\nu}}(x)\overline{\zeta^{\boldsymbol{\nu}}(y)}
\end{aligned}
$$

and

$$
C^{*}C\sum_{k=0}^{\infty}\sum_{\boldsymbol{\nu}\in\operatorname{supp}(c),|\boldsymbol{\nu}|=k} c_{\boldsymbol{\nu}}\zeta^{\boldsymbol{\nu}}(x)\overline{\zeta^{\boldsymbol{\nu}}(y)}=C^{*}\sum_{k=0}^{\infty}\sum_{\boldsymbol{\nu}\in\operatorname{supp}(c),|\boldsymbol{\nu}|=k} c_{\boldsymbol{\nu}}\overline{\zeta^{\boldsymbol{\nu}}(y)}=c_{\boldsymbol{\nu}} \qquad (5.19)
$$

From 5.18 and 5.19, we obtain that the only solution of the equation 5.17 is $c_0\begin{cases}0, & \varphi_0\neq 0;\\ 1, & \varphi_0=0.\end{cases}$ and $c_{\boldsymbol{\nu}}=\sum_{j=1}^{d}c_{\boldsymbol{\nu}-e_j}$ with $\boldsymbol{\nu}\neq 0$ which satisfies (5.12), that is f belongs to the fractional Drury-Arveson space $\mathcal{A}$.

□

As in the classical case we have the following corollary.

Corollary 5.9. *The multiplication operator M_{ζ_j} is a continuous operator in the fractional Drury-Arveson space and its adjoint is given by the Leibenson's shift operator R_j.*

Let s be a function such that the kernel

$$
k_s(x,y)=\sum_{k=0}^{\infty}\sum_{|\boldsymbol{\nu}|=k} c_{\boldsymbol{\nu}}\left(\zeta^{\boldsymbol{\nu}}(x)\overline{\zeta^{\boldsymbol{\nu}}(y)}-(\zeta^{\boldsymbol{\nu}}\otimes s)(x)\overline{(\zeta^{\boldsymbol{\nu}}\otimes s)(y)}\right)
$$

is positive. This function is called a Schur multiplier since it defines an operator M_s acting on a function f associated to the sequence $(f_{\boldsymbol{\nu}})_{\boldsymbol{\nu}}$ as

$$
M_s f=\sum_{\boldsymbol{\nu}}\zeta^{\boldsymbol{\nu}}\left(\sum_{|\boldsymbol{\mu}|\leq|\boldsymbol{\nu}|}s_{\boldsymbol{\mu}}f_{\boldsymbol{\nu}-\boldsymbol{\mu}}\right)
$$

and this operator is a contraction from ℓ^2 into the fractional Drury-Arveson space $\mathcal{A}$. One notes that M_s is a CK-multiplication of f with s from the left.

Now, we can write $k_s(\cdot, y)$ as

$$k_s(\cdot, y) = (I - M_s M_s^*) k_{\mathcal{A}}$$

and the right module operator range

$$\mathcal{H}(s) := (I - M_s M_s^*)^{\frac{1}{2}} \mathcal{A}$$

is the reproducing kernel module which is the counterpart to the de Branges-Rovnyak space in our setting. This operator range definition is one of the characterization of de Brange-Rovnyak spaces [19, 6].

Let $\ell^2(\mathbb{C}_d)$ denote the space of $\mathbb{C}_d-$valued sequences $\{(f_{\boldsymbol{\nu}})_{\boldsymbol{\nu}} : \boldsymbol{\nu} \in \mathbb{N}^d$ and $f_{\boldsymbol{\nu}} \in \mathbb{C}_d\}$ such that $\sum c_{\boldsymbol{\nu}} |f_{\boldsymbol{\nu}}|^2 < \infty$.

Theorem 5.10. *Given a $\mathcal{H}^*-$valued Schur multiplier s then there exists a co-isometry*

$$V = \begin{pmatrix} T_1 & F_1 \\ T_2 & F_2 \\ \vdots & \vdots \\ T_d & F_d \\ G & H \end{pmatrix} : \begin{pmatrix} \mathcal{H}(s) \\ \mathcal{H} \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{H}(s)^d \\ \mathcal{H} \end{pmatrix}$$

such that

$$f(x) - f(0) = \sum_{j=1}^{d} (\zeta_j \otimes T_j)(x);$$

$$(s(x) - s(0))\, h = \sum_{j=1}^{n} (\zeta_j \otimes F_j)(x);$$

$$Gf = f(0);$$

$$Hf = s(0)h.$$

Furthermore, $s(x)$ admits the representation

$$s(x)h = Hh + \sum_{k=1}^{d} \sum_{\boldsymbol{\nu} \in \mathbb{N}^d} c_{\boldsymbol{\nu}} (\zeta_j \otimes \zeta^{\boldsymbol{\nu}})(x) G T^{\boldsymbol{\nu}} F_k h, \quad x \in \Omega, h \in \mathcal{H},$$

where $T^{\boldsymbol{\nu}} := T_1^{\nu_1} \times \cdots \times T_d^{\nu_d}$.

Proof. We denote by $\mathcal{H}(s)_d$ the closure in $\mathcal{H}(s)^d$ of the linear span of the elements of the form

$$w_y = \begin{pmatrix} R_1 k_s(\cdot, y) \\ \vdots \\ R_d k_s(\cdot, y) \end{pmatrix} = \begin{pmatrix} (I - M_s M_s^*) R_1 k_{\mathcal{A}}(\cdot, y) \\ \vdots \\ (I - M_s M_s^*) R_n k_{\mathcal{A}}(\cdot, y) \end{pmatrix}, \quad y \in \Omega.$$

We define

$$(\hat{T} w_y q)(x) = (k_s(x, y) - k_s(x, 0))q, \quad (\hat{F} w_y q)(x) = (s(y)^* - s(0)^*)q$$

$$(\hat{G} q)(x) = k_s(x, 0)q, \quad \hat{H} q = s(0)^* q$$

keeping in mind the isometry

$$\left\langle \begin{pmatrix} \hat{T} w_{y_1} q_1 + \hat{G} p_1 \\ \hat{F} w_{y_1} q_1 + \hat{H} p_1 \end{pmatrix}, \begin{pmatrix} \hat{T} w_{y_2} q_2 + \hat{G} p_2 \\ \hat{F} w_{y_2} q_2 + \hat{H} p_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} w_{y_1} q_1 \\ p_1 \end{pmatrix}, \begin{pmatrix} w_{y_2} q_2 \\ p_2 \end{pmatrix} \right\rangle,$$

for any $y_1, y_2 \in \Omega$ and $p_1, p_2, q_1, q_2 \in \mathbb{C}_d$. The latter isometry is important for the definition of the operators since a priori a linear combination of w_{y_i} could be zero and correspond to a non-zero image. The isometry formula overcomes that problem since a densely defined isometric relation between Hilbert modules extends to the graph of an isometry. Hence, the operator matrix $\hat{V} = \begin{pmatrix} \hat{T} & \hat{G} \\ \hat{F} & \hat{H} \end{pmatrix}$ can be extended as an isometry from $\begin{pmatrix} \mathcal{H}(s) \\ \mathcal{H} \end{pmatrix}$ into $\begin{pmatrix} \mathcal{H}(s)^d \\ \mathcal{H} \end{pmatrix}$. Let us set $V = \begin{pmatrix} T & G \\ F & H \end{pmatrix} = \hat{V}^*$. Then the previous relations imply $f(x) - f(0) = \sum_{j=1}^d (\zeta_j \otimes T_j)(x), (s(x) - s(0))\, h = \sum_{j=1}^d (\zeta_j \otimes F_j)(x), Gf = f(0)$ and $Hf = s(0)h$. Now, iterating $f(x) - f(0) = \sum_{j=1}^d (\zeta_j \otimes T_j)(x)$ as before leads to the representation for $s(x)h$. □

Theorem 5.11. *Let $\mathcal{G}, \mathcal{H}$ be right $\mathbb{C}_d-$ Hilbert modules and let*

$$V = \begin{pmatrix} T_1 & F_1 \\ \vdots & \vdots \\ T_d & F_d \\ G & H \end{pmatrix} : \begin{pmatrix} \mathcal{G} \\ \mathcal{H} \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{G}^d \\ \mathbb{C}_d \end{pmatrix}$$

be a co-isometry. Then

$$s_V(x) = H + \sum_{k=1}^d \sum_{\boldsymbol{\nu} \in \mathbb{N}^d} c_{\boldsymbol{\nu}} (\zeta_j \otimes \zeta^{\boldsymbol{\nu}})(x) G T^{\boldsymbol{\nu}} F_k, \quad x \in \Omega,$$

is an $\mathcal{H}^*-$valued Schur multiplier.

Proof. Define

$$A_{\boldsymbol{\mu}}(x) = \sum_{\boldsymbol{\nu}\in\mathbb{N}^d} c_{\boldsymbol{\nu}} \left(\Pi_{j=1}^d(\zeta_j \otimes \zeta^{\boldsymbol{\nu}+\boldsymbol{\mu}})(x)GT^{\boldsymbol{\nu}}\right),$$

$$B_{\boldsymbol{\mu}}(x) = \sum_{\boldsymbol{\nu}\in\mathbb{N}^d} c_{\boldsymbol{\nu}}\ \zeta^{\boldsymbol{\nu}+\boldsymbol{\mu}}(x)GT^{\boldsymbol{\nu}}, \quad C(x) = \sum_{\boldsymbol{\nu}\in\mathbb{N}^d} c_{\boldsymbol{\nu}}\ \zeta^{\boldsymbol{\nu}}(x)GT^{\boldsymbol{\nu}}.$$

We get

$$\begin{aligned}
A_{\boldsymbol{\mu}}(x)A_{\boldsymbol{\mu}}(y)^* + \zeta^{\boldsymbol{\nu}}(x)\overline{\zeta^{\boldsymbol{\nu}}(y)} &= \left(A_{\boldsymbol{\mu}}(x)\zeta^{\boldsymbol{\nu}}(x)\right)VV^*\left(A_{\boldsymbol{\mu}}(y)\zeta^{\boldsymbol{\nu}}(y)\right)^* \\
&= \left[B_{\boldsymbol{\mu}}(x)(\zeta^{\boldsymbol{\nu}}\otimes s_V)(x)\right]\left[B_{\boldsymbol{\mu}}(y)(\zeta^{\boldsymbol{\nu}}\otimes s_V)(\mathbf{y})\right]^* \\
&= B_{\boldsymbol{\mu}}(x)B_{\boldsymbol{\mu}}(y)^* + (\zeta^{\boldsymbol{\nu}}\otimes s_V)(x)\left[(\zeta^{\boldsymbol{\nu}}\otimes s_V)(y)\right]^*.
\end{aligned}$$

Hence

$$\begin{aligned}
k_{s_V}(x,y) &= \sum_{\boldsymbol{\mu}\in\mathbb{N}^n} c_{\boldsymbol{\mu}}\left(\zeta^{\boldsymbol{\mu}}(x)\overline{\zeta^{\boldsymbol{\mu}}(y)} - (\zeta^{\boldsymbol{\nu}}\otimes s_V)(x)(\zeta^{\boldsymbol{\nu}}\otimes s_V)(y)^*\right) \\
&= \sum_{\boldsymbol{\mu}\in\mathbb{N}^n} c_{\boldsymbol{\mu}}\left(B_{\boldsymbol{\mu}}(x)B_{\boldsymbol{\mu}}(y)^* - A_{\boldsymbol{\mu}}(x)A_{\boldsymbol{\mu}}(y)^*\right).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\sum_{\boldsymbol{\mu}\in\mathbb{N}^d} c_{\boldsymbol{\mu}} \left(\sum_{j=1}^d(\zeta^{\boldsymbol{\nu}+\boldsymbol{\mu}}\otimes\zeta_j)(x)GT^{\boldsymbol{\nu}}(T^{\boldsymbol{\eta}})^*G^*\overline{\zeta^{\boldsymbol{\nu}+\boldsymbol{\mu}}\otimes\zeta_j)(y)}\right) \\
&= \sum_{j=1}^d\sum_{\boldsymbol{\mu}:\mu_n>0}\frac{\mu_n}{|\boldsymbol{\mu}|}\ c_{\boldsymbol{\mu}}\ \zeta^{\boldsymbol{\nu}+\boldsymbol{\mu}}(x)GT^{\boldsymbol{\nu}}(T^{\boldsymbol{\eta}})^*G^*\overline{\zeta^{\boldsymbol{\eta}+\boldsymbol{\mu}}(\mathbf{y})} \\
&= \sum_{\boldsymbol{\mu}:|\boldsymbol{\mu}|>0}\zeta^{\boldsymbol{\nu}+\boldsymbol{\mu}}(x)GT^{\boldsymbol{\nu}}(T^{\boldsymbol{\eta}})^*G^*\overline{\zeta^{\boldsymbol{\eta}+\boldsymbol{\mu}}(y)},
\end{aligned}$$

so that $k_{s_V}(x,y) = C(x)C(y)^*$. □

5.2 The Case of the Ternary Clifford Algebras

The aim of this section is to present the Cauchy-Kovalevskaya extension for the case of ternary Clifford algebras. To this effect, we first recall the definition of the ternary Clifford algebra $\mathcal{C}\ell_d^{1/3}$ which is the algebra generated by $\{e_1,\cdots,e_d\}$ subject to the multiplication rule:

$$e_i^3 = 1, \quad e_ie_j = \omega e_je_i, \text{ for } 1 \le i < j \le d, \tag{5.20}$$

where $\omega = e^{i2\pi/3}$.

We consider the fractional Cauchy-Riemann operator

$$D_\varphi = \sum_{j=0}^{d} e_j \partial_j^\varphi = e_0\partial_0^\varphi + e_1\partial_1^\varphi + \cdots + e_d\partial_d^\varphi, \tag{5.21}$$

where ∂_j^α represents the Gelfond-Leontiev generalized derivative with respect to the $j-$coordinate and the entire function φ with order $\rho > 0$ and type $\sigma > 0$.

5.2.1 Ternary Cauchy-Kovalevskaya Extension

Theorem 5.12 (Ternary Cauchy-Kovalevskaya extension)**.** *Let $P_{\boldsymbol{\nu}}$ be a homogeneous product given by $P_{\boldsymbol{\nu}}(x_1,\cdots,x_d) := x_1^{\nu_1}\cdots x_d^{\nu_d}$, $\boldsymbol{\nu} = (\nu_1,\cdots,\nu_d) \in \mathbb{N}_0^d$ and let φ be an entire function with order $\rho > 0$ and type $\sigma \neq 0$. The Cauchy-Kovalevskaya extension $CK_\varphi[P_{\boldsymbol{\nu}}]$ of $P_{\boldsymbol{\nu}}$ has the form*

$$CK_\varphi[P_{\boldsymbol{\nu}}](x_0,x_1,\cdots,x_d) := \left[\varphi\left(-x_0e_0^2\underline{D}_\varphi\right)\right] x_1^{\nu_1}\cdots x_d^{\nu_d}. \tag{5.22}$$

Moreover, $CK_\varphi[P_{\boldsymbol{\nu}}]$ is a monogenic polynomial homogeneous of degree $|\boldsymbol{\nu}| := \nu_1+\cdots+\nu_d$ and $\underline{D}_\varphi := \sum_{j=1}^d e_j\partial_j^\varphi$ with ∂_j^φ is the Gelfond-Leontiev operator of generalized differentiation with respect to the function φ.

Proof. Using $D_\varphi = e_0\partial_0^\varphi + \underline{D}_\varphi$ we have

$$\begin{aligned}
D_\varphi CK_\varphi[P_{\boldsymbol{\nu}}](x_0,x_1,\cdots,x_d) &= D_\varphi\left[\varphi\left(-x_0e_0^2\underline{D}_\varphi\right)\right] x_1^{\nu_1}\cdots x_d^{\nu_d}\\
&= (e_0\partial_0^\varphi + \underline{D}_\varphi)\sum_{k=0}^{\infty}\varphi_k(-1)^k(x_0e_0^2\underline{D}_\varphi)^k(x_1^{\nu_1}\cdots x_d^{\nu_d})\\
&= \sum_{k=1}^{\infty}\varphi_k(-1)^k\frac{\varphi_{k-1}}{\varphi_k}(x_0)^{k-1}e_0(e_0^2\underline{D}_\varphi)^k(x_1^{\nu_1}\cdots x_d^{\nu_d}) + \sum_{k=0}^{\infty}\varphi_k(-1)^k\underline{D}_\varphi(x_0e_0^2\underline{D}_\varphi)^k(x_1^{\nu_1}\cdots x_d^{\nu_d})\\
&= \sum_{k=1}^{\infty}\varphi_{k-1}(-1)^k e_0(e_0^2\underline{D}_\varphi)(x_0e_0^2\underline{D}_\varphi)^{k-1}(x_1^{\nu_1}\cdots x_d^{\nu_d}) + \sum_{k=0}^{\infty}\varphi_k(-1)^k\underline{D}_\varphi(x_0e_0^2\underline{D}_\varphi)^k(x_1^{\nu_1}\cdots x_d^{\nu_d})\\
&= \sum_{k=0}^{\infty}\varphi_k(-1)^{k+1}\underline{D}_\varphi(x_0e_0^2\underline{D}_\varphi)^k(x_1^{\nu_1}\cdots x_d^{\nu_d}) + \sum_{k=0}^{\infty}\varphi_k(-1)^k\underline{D}_\varphi(x_0e_0^2\underline{D}_\varphi)^k(x_1^{\nu_1}\cdots x_d^{\nu_d})\\
&= 0.
\end{aligned}$$

□

Now, we define the corresponding Fueter polynomials in the ternary setting.

Definition 5.9. *Let $\boldsymbol{\nu} = (\nu_1,\cdots,\nu_d) \in \mathbb{N}_0^d$ with $|\boldsymbol{\nu}| = \nu_1+\cdots+\nu_d$. Then the fractional monogenic powers*

$$\zeta^{\boldsymbol{\nu}}(x) = CK_\varphi[P_{\boldsymbol{\nu}}](x_0,x_1,\ldots,x_d) := \sum_{j=0}^{|\boldsymbol{\nu}|}(-1)^j\varphi_j(x_0e_0^2\underline{D}_\varphi)^j(x_1^{\nu_1}\cdots x_d^{\nu_d}) \tag{5.23}$$

are called the ternary fractional Fueter polynomials of degree $|\boldsymbol{\nu}|$.

Example 5.4. *We have* $\zeta^0(x) = \varphi_0$*, while for* $\boldsymbol{\nu} = (0, \cdots, \nu_j, \cdots, 0)$ *with* $\nu_j = 1$*, that is,* $\boldsymbol{\nu} = e_j$

$$\zeta^{e_j}(x) = \sum_{j=0}^{1} (-1)^j \varphi_j (x_0 e_0^2 \underline{D}_{\varphi})^j (x_1^0 \cdots x_j^1 \cdots x_d^0)$$
$$= \varphi_0 x_j - \varphi_1 x_0 e_0^2 \frac{\varphi_0}{\varphi_1} e_j = \varphi_0 (x_j - x_0 e_0^2 e_j) \quad j = 1, \ldots, d.$$

Example 5.5. *We consider the case of* $d = 3$ *and the Mittag-Leffer function* $\varphi(z) = E_{\frac{2}{3},1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+2k/3)}$*. By using the ternary Cauchy-Kovalevskaya extension, we can find the basis for the space of fractional homogeneous monogenic polynomials* $\mathcal{M}_l$ *with* $l = 1, 2, 3$*. In this case, we obtain the two polynomials which are the basis for the space* $\mathcal{M}_1$*:*

$$\begin{aligned} V_1^{1,\frac{2}{3}}\left(x^{\frac{2}{3}}\right) &= -e_0^2 e_1\, x_0 + x_1, \\ V_2^{1,\frac{2}{3}}\left(x^{\frac{2}{3}}\right) &= -e_0^2 e_2\, x_0 + x_2; \end{aligned}$$

the three polynomials which are the basis for the space $\mathcal{M}_2$*:*

$$\begin{aligned} V_1^{2,\frac{2}{3}}\left(x^{2/3}\right) &= \omega\ e_0 e_1^2\ x_0^2 - \frac{\Gamma\left(\frac{7}{3}\right)}{\Gamma^2\left(\frac{5}{3}\right)}\ e_0^2 e_1\ x_0\ x_1 + x_1^2, \\ V_2^{2,\frac{2}{3}}\left(x^{2/3}\right) &= -\frac{\Gamma^2\left(\frac{5}{3}\right)}{\Gamma\left(\frac{7}{3}\right)}\ \omega\ e_0^2 e_2^2\ x_0^2 - e_0^2 e_2\ x_0\ x_1 - e_0^2 e_1\ x_0\ x_2 \\ &\quad + x_1\ x_2, \\ V_3^{2,\frac{2}{3}}\left(x^{2/3}\right) &= \omega\ e_0 e_2^2\ x_0^2 \\ &\quad - \frac{\Gamma\left(\frac{7}{3}\right)}{\Gamma^2\left(\frac{5}{3}\right)}\ e_0^2 e_2\ x_0\ x_2 + x_2^2. \end{aligned}$$

and the four polynomials which are the basis for the space $\mathcal{M}_3$:

$$\begin{aligned}
V_1^{3,\frac{2}{3}}\left(x^{2/3}\right) &= -x_0^3 + \frac{27\sqrt{3}}{8\pi}\,\omega\, e_0 e_1^2\, x_0^2\, x_1 - \frac{27\sqrt{3}}{8\pi}\, e_0^2 e_1\, x_0\, x_1^2 + x_1^3,\\
V_2^{3,\frac{2}{3}}\left(x^{2/3}\right) &= -e_1^2\, x_0^2\, x_1 + e_2^2\, x_0^2\, x_2 - e_0 e_1\, x_0\, x_1^2\\
&\quad -\frac{\Gamma\left(\frac{7}{3}\right)}{\Gamma^2\left(\frac{5}{3}\right)}\, e_2\, x_0\, x_1\, x_2 + x_1^2\, x_2,\\
V_3^{3,\frac{2}{3}}\left(x^{2/3}\right) &= \omega\, e_1 e_2\, x_0^2\, x_1 - e_1^2\, x_0^2\, x_2\\
&\quad -\frac{\Gamma\left(\frac{7}{3}\right)}{\Gamma^2\left(\frac{5}{3}\right)}\, e_0^2 e_2\, x_0\, x_1\, x_2 - e_2\, x_0\, x_2^2 + x_1\, x_2^2,\\
V_4^{3,\frac{2}{3}}\left(x^{2/3}\right) &= -x_0^3 + \frac{27\sqrt{3}}{8\pi}\,\omega\, e_0 e_2^2\, x_0^2\, x_2\\
&\quad -\frac{27\sqrt{3}}{8\pi}\, e_0^2 e_2\, x_0\, x_2^2 + x_2^3.
\end{aligned}$$

We remark that this last example has the same elements of the basis in the Example (4.5) which is used $e_2 = e_1 e_3$.

Therefore, it follows the next statement.

Theorem 5.13. *If $P_{\boldsymbol{\nu}}(x_1,\cdots,x_d) := x_1^{\nu_1}\cdots x_d^{\nu_d}$, $\boldsymbol{\nu} = (\nu_1,\cdots,\nu_d) \in \mathbb{N}_0^d$ is a homogeneous product then the set $\{\zeta^{\boldsymbol{\nu}} | \zeta^{\boldsymbol{\nu}}(x) = CK_\varphi[P_{\boldsymbol{\nu}}]\}$ is a basis for the space of fractional homogeneous monogenic polynomials $\mathcal{M}_l$.*

As in the case of Clifford algebras, we can obtain the Cauchy-Kovalevskaya extension for $\mathbb{R}_d$-valued real analytic function in the ternary setting since real analytic function near the origin admits a decomposition into homogeneous polynomials.

Moreover, by using the fractional ternary Fueter polynomials (5.23) we may define a corresponding reproducing kernel right-Hilbert module so that we obtain the solution for Gleason's problem in this setting. The ternary fractional monogenic operator-valued functions are presented as well as its reproducing kernel spaces are described.

In what follows we will see the corresponding definitions and results in the ternary setting which are very similar to the case of classic Clifford algebras described in the first section of this chapter. Proof of the corresponding results will be omitted since it is identical to the classical case.

5.2.2 Reproducing Kernel Right-Hilbert Module and the Gleason's Problem

We start this subsection with the definition of the right Clifford module of fractional power series in the ternary algebra setting as follows.

Definition 5.10. *Let $\zeta^{\boldsymbol{\nu}}$ be the fractional monogenic powers defined in (5.23). Then the right Clifford module $\mathcal{M}$ of the space these fractional monogenic powers is defined by*

$$f(x) = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \zeta^{\boldsymbol{\nu}}(x) f_{\boldsymbol{\nu}} = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} CK_{\varphi}[P_{\boldsymbol{\nu}}](x_0, x_1, \cdots, x_d) f_{\boldsymbol{\nu}}, \tag{5.24}$$

where $\sum_{\boldsymbol{\nu}} |f_{\boldsymbol{\nu}}|^2 < \infty$.

This series (5.24) is called ternary fractional Fueter series and it is easy to see that $Df = 0$ for all $f \in \mathcal{M}$.

Let $(\partial_{\varphi})^{\boldsymbol{\mu}} = (\partial_1^{\varphi})^{\mu_1} \cdots (\partial_d^{\varphi})^{\mu_d}$ which acts on elements $x^{\boldsymbol{\nu}} = x_1^{\nu_1} \cdots x_d^{\nu_d}$ as

$$(\partial_{\varphi})^{\boldsymbol{\mu}}(x_1^{\nu_1} \cdots x_d^{\nu_d}) \quad = \quad \Pi_{j=1}^{k} \varphi(\nu_j, \nu_j - \mu_j) x_1^{\nu_1-\mu_1} \cdots x_d^{\nu_d-\mu_d},$$

where $\varphi(\nu_j, \nu_j - \mu_j) = \begin{cases} \dfrac{\varphi_{\nu_j-\mu_j}}{\varphi_{\nu_j}}, & \nu_j \geq \mu_j; \\ 0, & \nu_j < \mu_j \ . \end{cases}$ and ∂_i^{φ} $(i = 1, \cdots, d)$ is the GL operator of generalized differentiation with respect to x_i.

Lemma 5.14. *Given a function f defined in (5.24) we have*

$$(\partial_{\varphi})^{\boldsymbol{\mu}} f(x) = \sum_{l=0}^{\infty} \sum_{|\boldsymbol{\nu}|=l} \varphi(\boldsymbol{\nu}, \boldsymbol{\nu} - \boldsymbol{\mu}) \zeta^{\boldsymbol{\nu}-\boldsymbol{\mu}}(x) f_{\boldsymbol{\nu}},$$

where $\varphi(\boldsymbol{\nu}, \boldsymbol{\nu} - \boldsymbol{\mu}) = \begin{cases} \Pi_{j=1}^{d} \varphi(\nu_j, \nu_j - \mu_j) = \Pi_{j=1}^{d} \dfrac{\varphi_{\nu_j-\mu_j}}{\varphi_{\nu_j}}, & \nu_j \geq \mu_j, \ j = 1, \cdots, d; \\ 0, & \exists j; \ \nu_j < \mu_j. \end{cases}$

Thus, we define the Clifford-valued coefficients $f_{\boldsymbol{\nu}}$ as

$$f_{\boldsymbol{\nu}} := \frac{1}{\varphi(\boldsymbol{\nu}, 0)\varphi_0} \left. (\partial_{\varphi})^{\boldsymbol{\nu}} f(x) \right|_{x=0}, \tag{5.25}$$

which allows to define the Cauchy product.

Definition 5.11. *Let $f, g \in \mathcal{M}$ whose series expansion are given by*

$$f(x) = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \zeta^{\boldsymbol{\nu}}(x) f_{\boldsymbol{\nu}}, \qquad g(x) = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\mu}|=k} \zeta^{\boldsymbol{\mu}}(x) g_{\boldsymbol{\mu}}.$$

We define a Cauchy product between f and g as

$$(f \otimes g)(x) := \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \zeta^{\boldsymbol{\nu}}(x) \left(\sum_{0 \leq |\boldsymbol{\mu}| \leq |\boldsymbol{\nu}|} f_{\boldsymbol{\mu}} g_{\boldsymbol{\nu}-\boldsymbol{\mu}} \right). \tag{5.26}$$

In particular, if $f(x) = \zeta^{e_i}(x)$ and $g(x) = \zeta^{e_j}(x)$ $(i \neq j)$, the Cauchy product between f and g is given by

$$(\zeta^{e_i} \otimes \zeta^{e_j})(x) = \zeta^{e_i+e_j}(x) = (\zeta^{e_j} \otimes \zeta^{e_i})(x).$$

and consequently, $(\zeta^{e_j} \otimes \zeta^{\boldsymbol{\nu}-e_j})(x) = \zeta^{\boldsymbol{\nu}}(x)$.

In order to present the reproducing kernel right-Hilbert module (RKHS) which arises from the monogenic formal powers defined in (5.23), as in the case of Clifford algebras we need some previous definitions.

Definition 5.12. *Given a sequence $c = (c_{\boldsymbol{\nu}}),\ c_{\boldsymbol{\nu}} \geq 0$, for all $\boldsymbol{\nu} \in \mathbb{N}^d$, we define its support as*

$$\operatorname{supp}(c) := \{\boldsymbol{\nu} \in \mathbb{N}^d : c_{\boldsymbol{\nu}} \neq 0\}. \tag{5.27}$$

We now consider the domain

$$\Omega_c := \left\{ x = x_0 + x_1e_1 + \cdots + x_de_d \in \mathbb{R}^{n+1} : \sum_{k=0}^{\infty} \sum_{\boldsymbol{\nu}\in\operatorname{supp}(c), |\boldsymbol{\nu}|=k} c_{\boldsymbol{\nu}}|\zeta^{\boldsymbol{\nu}}(x)|^2 < \infty \right\} \tag{5.28}$$

and then we define the kernel

$$k_c(x,y) = \sum_{k=0}^{\infty} \sum_{\boldsymbol{\nu}\in\operatorname{supp}(c), |\boldsymbol{\nu}|=k} c_{\boldsymbol{\nu}}\zeta^{\boldsymbol{\nu}}(x)\overline{\zeta^{\boldsymbol{\nu}}(y)}$$

associated to the domain Ω_c.

Given the associated reproducing kernel right-Hilbert module $\mathcal{H}(c)$ we say that the function $f(x) = \sum_{\boldsymbol{\nu}\in\operatorname{supp}(c)} \zeta^{\boldsymbol{\nu}}(x) f_{\boldsymbol{\nu}}$ belongs to $\mathcal{H}(c)$ if it satisfies $\|f\|_c^2 := \sum_{\boldsymbol{\nu}\in\operatorname{supp}(c)} \frac{|f_{\boldsymbol{\nu}}|^2}{c_{\boldsymbol{\nu}}} < \infty$ and thus, we can define the reproducing formula $f(x) = \int_{\Omega_c} k_c(x,y)f(y)dy$ which preserves right-linearity with respect to Clifford-valued constants.

From (5.25), by taking $\boldsymbol{\mu} = e_j = (0, \cdots, 1, \cdots, 0)$ for $j = 1, \cdots, d$, it follows that

$$(\partial_j^{\varphi})^{e_j}\zeta^{\boldsymbol{\nu}}(x) \quad = \quad \varphi(\nu_j, \nu_j - 1)\zeta^{\boldsymbol{\nu}-e_j}(x)$$

where $\boldsymbol{\nu} - e_j = (\boldsymbol{\nu}_1, \cdots, \boldsymbol{\nu}_j - 1, \cdots, \boldsymbol{\nu}_d)$ and $\boldsymbol{\nu}_j \geq 1$.

Given $|\varphi(\boldsymbol{\nu}, \boldsymbol{\nu} - 1)| = \displaystyle\sum_{j=1}^{d} \varphi(\nu_j, \nu_j - 1)$ we obtain the corresponding Leibenson's shift operator $R_j : \mathcal{H}(c) \to \mathcal{H}(c)$ which is defined by

$$R_j f(x) = \sum_{k=1}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \zeta^{\boldsymbol{\nu}-e_j}(x) \frac{\varphi(\nu_j, \nu_j - 1)}{|\varphi(\boldsymbol{\nu}, \boldsymbol{\nu} - 1)|} f_{\boldsymbol{\nu}}. \tag{5.29}$$

As before, this operator is bounded in $\mathcal{H}(c)$ iff the set $\operatorname{supp}(c)$ is lower inclusive and it holds that

$$\sup_j \left| \frac{\varphi(\nu_j, \nu_j - 1)}{|\varphi(\boldsymbol{\nu}, \boldsymbol{\nu} - 1)|} \right|^2 \frac{c_{\boldsymbol{\nu}}}{c_{\boldsymbol{\nu} - e_j}} \quad < \quad \infty. \tag{5.30}$$

The version of Gleason's problem is given as following.

Definition 5.13 (Gleason's problem)**.** *Let $\mathcal{M}$ be a set of $\mathcal{C}\ell_d^{1/3}$−valued functions which are fractional monogenic in a neighbourhood of the origin and let $\boldsymbol{\nu} = e_j = (0, \cdots, 1, \cdots, 0)$. The Gleason's problem for $f \in \mathcal{M}$ is to find functions $p_1, \ldots, p_d \in \mathcal{M}$ such that*

$$f(x) - f(0) = \sum_{j=1}^{d} (\zeta^{e_j} \otimes p_j)(x). \tag{5.31}$$

It turns out that the Leibenson's shift operators (5.29) provide a solution for this new Gleason's problem as we can see in the following theorem.

Theorem 5.15. *Gleason's problem is solvable in the reproducing kernel right-Hilbert module $\mathcal{H}(c)$ where the weights c satisfy (5.30) and our Leibenson's shift operators provide the only commutative solution of the problem.*

5.2.3 Fractional Monogenic Operator-Valued Functions and Reproducing Kernel Spaces

Let $\mathcal{H}$ be a right-Clifford Hilbert module with the sesqui-linear form $\langle f, g \rangle = \int_{\mathbb{R}^{d+1}} f(x)\overline{g(x)}dx$. Then its dual space $\mathcal{H}^*$ is also a right-linear Clifford Hilbert module.

Definition 5.14. *Let $\Omega \subset \mathbb{R}^{d+1}$ be a domain containing the origin. A mapping $f : \Omega \mapsto \mathcal{H}^*$ is called an $\mathcal{H}^*$-valued (left-) monogenic function in Ω if for all $h \in \mathcal{H}$ we have $f(\cdot)h$ being a (left-) monogenic function in Ω.*

It follows the following theorem.

Theorem 5.16. *Let f be an $\mathcal{H}^*$-valued monogenic function in a ball $B(0, R)$ centered at the origin with radius R. Then f has a representation via*

$$f(x) = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \zeta^{\boldsymbol{\nu}}(x) f_{\boldsymbol{\nu}}$$

where $f_{\boldsymbol{\nu}} \in \mathcal{H}^*$ are linear functionals over $\mathcal{H}$. Hereby, the series converges normally in $B(0, R)$.

Corollary 5.17. *A reproducing kernel $k(x, y)$ can be represented as $k(x, y) = g(x)g(y)^*$, where $g(x)$ is an $\mathcal{H}(k)^*$-valued monogenic function and $\mathcal{H}(k)^*$ denotes the dual of the RKHS associated to $k = k(\cdot, \cdot)$.*

We now introduce a reproducing kernel Hilbert space of fractional monogenic functions in a domain Ω_c in $\mathbb{R}^{d+1}$.

Definition 5.15. *We define the fractional Drury-Arveson space (or module) $\mathcal{A}$ as the RKHS with the reproducing kernel $k_c(x,y)$ given by*

$$k_{\mathcal{A}}(x,y) = \sum_{k=0}^{\infty} \sum_{\boldsymbol{\nu}\in \mathrm{supp}(c), |\boldsymbol{\nu}|=k} c_{\boldsymbol{\nu}} \zeta^{\boldsymbol{\nu}}(x)\overline{\zeta^{\boldsymbol{\nu}}(y)},$$

where the coefficients $c = c_{\boldsymbol{\nu}}$ satisfy (5.30).

Theorem 5.18. *Let C be the operator of evaluation at the origin, that is*

$$Cf := \frac{1}{\varphi(\boldsymbol{\nu},0)\varphi_0} \left((\partial^{\alpha})^{\boldsymbol{\nu}} f(x)\right)|_{x=0}$$

and let M_{ζ_j} be the multiplication operator $M_{\zeta_j} f = f \otimes \zeta_j$. Then we have

$$(I - \sum_{j=1}^{d} M_{\zeta_j} M_{\zeta_j}^*) = C^* C \tag{5.32}$$

if and only if f belongs to the fractional Drury-Arveson space $\mathcal{A}$.

As in the classical case we have the following corollary.

Corollary 5.19. *The multiplication operator M_{ζ_j} is a continuous operator in the fractional Drury-Arveson space and its adjoint is given by the Leibenson's shift operator R_j.*

Let s be a function such that the kernel

$$k_s(x,y) = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} c_{\boldsymbol{\nu}} \left(\zeta^{\boldsymbol{\nu}}(x)\overline{\zeta^{\boldsymbol{\nu}}(y)} - (\zeta^{\boldsymbol{\nu}} \otimes s)(x)\overline{(\zeta^{\boldsymbol{\nu}} \otimes s)(y)}\right)$$

is positive. This function is called a Schur multiplier since it defines an operator M_s acting on a function f associated to the sequence $(f_{\boldsymbol{\nu}})_{\boldsymbol{\nu}}$ as

$$M_s f = \sum_{\boldsymbol{\nu}} \zeta^{\boldsymbol{\nu}} \left(\sum_{|\boldsymbol{\mu}|\leq|\boldsymbol{\nu}|} s_{\boldsymbol{\mu}} f_{\boldsymbol{\nu}-\boldsymbol{\mu}} \right)$$

and this operator is a contraction from ℓ^2 into the fractional Drury-Arveson space $\mathcal{A}$. One notes that M_s is a CK-multiplication of f with s from the left.

Now, we can write $k_s(\cdot, y)$ as

$$k_s(\cdot,y) = (I - M_s M_s^*) k_{\mathcal{A}}$$

and the right module operator range

$$\mathcal{H}(s) := (I - M_s M_s^*)^{\frac{1}{2}} \mathcal{A}$$

is the reproducing kernel module which is the counterpart to the de Branges-Rovnyak space in our setting.

Let $\ell^2(\mathbb{C}_d)$ denote the space $\mathbb{C}_d$−valued sequences $\{(f_{\boldsymbol{\nu}})_{\boldsymbol{\nu}} : \boldsymbol{\nu} \in \mathbb{N}^d \text{ and } f_{\boldsymbol{\nu}} \in \mathbb{C}_d\}$ such that $\sum c_{\boldsymbol{\nu}} |f_{\boldsymbol{\nu}}|^2 < \infty$.

Theorem 5.20. *Given a $\mathcal{H}^*$−valued Schur multiplier s then there exists a co-isometry*

$$V = \begin{pmatrix} T_1 & F_1 \\ T_2 & F_2 \\ \vdots & \vdots \\ T_d & F_d \\ G & H \end{pmatrix} : \begin{pmatrix} \mathcal{H}(s) \\ \mathcal{H} \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{H}(s)^d \\ \mathcal{H} \end{pmatrix}$$

such that

$$f(x) - f(0) = \sum_{j=1}^{d} (\zeta_j \otimes T_j)(x);$$

$$(s(x) - s(0))\, h = \sum_{j=1}^{n} (\zeta_j \otimes F_j)(x);$$

$$Gf = f(0);$$

$$Hf = s(0)h.$$

Furthermore, $s(x)$ admits the representation

$$s(x)h = Hh + \sum_{k=1}^{d} \sum_{\boldsymbol{\nu} \in \mathbb{N}^d} c_{\boldsymbol{\nu}} (\zeta_j \otimes \zeta^{\boldsymbol{\nu}})(x) G T^{\boldsymbol{\nu}} F_k h, \quad x \in \Omega, h \in \mathcal{H},$$

where $T^{\boldsymbol{\nu}} := T_1^{\nu_1} \times \cdots \times T_d^{\nu_d}$.

Theorem 5.21. *Let $\mathcal{G}, \mathcal{H}$ be right Hilbert modules over $\mathbb{C}_d$ and let*

$$V = \begin{pmatrix} T_1 & F_1 \\ \vdots & \vdots \\ T_d & F_d \\ G & H \end{pmatrix} : \begin{pmatrix} \mathcal{G} \\ \mathcal{H} \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{G}^d \\ \mathbb{C}_d \end{pmatrix}$$

be a co-isometry. Then

$$s_V(x) = H + \sum_{k=1}^{d} \sum_{\boldsymbol{\nu} \in \mathbb{N}^d} c_{\boldsymbol{\nu}} (\zeta_j \otimes \zeta^{\boldsymbol{\nu}})(x) G T^{\boldsymbol{\nu}} F_k, \quad x \in \Omega,$$

is an $\mathcal{H}^*-$*valued Schur multiplier.*

Chapter 6

Integral Transforms for Fractional Derivatives

The classic Fourier transform is defined via integration against the bounded eigenfunctions of the classic derivatives over $\mathbb{R}$. Another, more specific view is to note that the classic Laplacian over $L_2(\mathbb{R})$ is a selfadjoint operator and, therefore, has a real spectrum. The Fourier transform can be seen as the spectral decomposition of the function with respect to the Laplacian. This observation leads us to the question of Fourier-like integral transforms over $L_2(\mathbb{R})$. The problem is that in general neither fractional derivatives nor the Laplacian built by the second order fractional derivatives (like the one we considered in the previous chapters) are self-adjoint operators which makes a theory solemnly built on them difficult to handle. To get Fourier-like integral transforms it is better to use the Laplacian built by the derivatives and their adjoints which is self-adjoint by construction.

In this chapter we will present the building blocks of a theory for Fourier-like integral transforms linked to fractional derivative operators with respect to Gelfond-Leontiev operator of generalized differentiation. To this end, we need to calculate the adjoint operator to a Gelfond-Leontiev operators of generalized differentiation with respect to the inner product of $L_2(\mathbb{R})$. To determine the adjoint we will first prove that the Gelfond-Leontiev derivative operator can be written as a series expansion of the standard derivatives. This allows us to provide a general formula for the determination of the adjoint operator and, consequently, the resulting Laplace operator. We will find an explicit formula for the Gelfond-Leontiev Laplace operator and its generalized eigenfunctions which are used to define the Fourier-like transform associated to the Gelfond-Leontiev derivative operator.

Obviously, such an integral transform allows the definition of the convolution as the pre-image of the pointwise product of the transforms. In this context, the corresponding properties for convolution operator are presented and proved.

6.1 Gelfond-Leontiev Laplace Operator

In this section, we will obtain a representation of the Gelfond-Leontiev derivative operator as a series of standard derivatives which allows us to find the formal representation for its adjoint derivative operator. The explicit formula for the Gelfond-Leontiev Laplace operator is given as well as their generalized eigenfunctions.

6.1.1 Adjoint Gelfond-Leontiev Derivative Operator

Given a fractional operator F^α in some domain we denote $(F^\alpha)^*$ as its adjoint fractional operator. In [3, 18] we can see an example where it is showed that, for $L_{1,loc}[0,\infty)$, the Riemann-Liouville integral operator of order $\alpha(\alpha > 0)$ defined as

$$I^\alpha_{0+} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \quad f \in L_{1,loc}[0,\infty)$$

and the Weyl integral operator of order $\alpha(\alpha > 0)$ which is given by

$${}^W I^\alpha_+ f(x) = \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \frac{f(t)}{(x-t)^{1-\alpha}} dt \quad f \in L_{1,loc}(0,\infty)$$

are adjoint to each other.

It means that if we can find a function space Φ which is mapped by one of the operators I^α_{0+}, ${}^W I^\alpha_+$ continuously into another function space Ψ, then the dual or adjoint operator will map generalized functions on Ψ (as elements of the the dual space Ψ') into generalized functions on Φ. Thus, this adjoint operator will define the other of I^α_{0+}, ${}^W I^\alpha_+$ for generalized functions. Furthermore, in the case of a Hilbert space like $L_2(\mathbb{R})$ we know that the dual space coincides with the original space by Riesz representation theorem which allows us to define a self-adjoint second order operator by combining the adjoint operator with the original fractional derivative. This will be our approach to the Gelfond-Leontiev Laplacian.

We now consider the fractional derivative operator D^*_α as an adjoint fractional differential operator to the fractional differential operator D_α and $\mathcal{F}(\Omega)$ as the space of analytic complex-valued functions $f(z)$ in a disk $|z| < R$ for some $R > 0$. Then we can define the associated fractional Laplacian of order α as follows.

Definition 6.1. *Given a fractional derivative $D_\alpha : \mathcal{F}(\Omega) \to \mathcal{F}(\Omega)$ we define the associated fractional Laplacian of order α as*

$$\Delta_\alpha f = D^*_\alpha D_\alpha f, \tag{6.1}$$

where D^*_α denotes the adjoint operator of D_α.

In order to obtain the formula for the GL Laplace operator, we will prove that the GL derivative can be expressed by an interesting series of the standard derivatives as we can see in the next lemma.

Lemma 6.1. *Given an entire function $\varphi(z) = \sum_{k=0}^{\infty} \varphi_k \, z^k$ with order $\rho > 0$ and degree $\sigma > 0$, the GL fractional derivative, D_φ, can be written as*

$$D_\varphi f(z) = \sum_{k=1}^{\infty} a_k z^{k-1} \partial^k f(z). \tag{6.2}$$

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n \, z^n$. Since the GL differentiation operator D_φ with respect to the function φ is the linear operator which acts on powers of z^n as

$$D_\varphi z^n := \begin{cases} 0, & n = 0; \\ \frac{\varphi_{n-1}}{\varphi_n} z^{n-1}, & n = 1, 2, \cdots. \end{cases},$$

where φ is an entire function with order $\rho > 0$ and type $\sigma \neq 0$ which satisfies

$$\lim_{k\to\infty} \sqrt[k]{\left|\frac{\varphi_{k-1}}{\varphi_k}\right|} = 1 \tag{6.3}$$

and $\partial^k z^n = n(n-1)\cdots(n-k+1)z^{n-k} = \frac{n!}{(n-k)!} z^{n-k}$, formula (6.2) is equivalent to solve the following system

$$\begin{aligned} \frac{\varphi_{n-1}}{\varphi_n} z^{n-1} &= D_\varphi z^n \\ &= \sum_{k=1}^{\infty} a_k z^{k-1} (\partial^k z^n) \\ &= \sum_{k=1}^{n} \frac{n!}{(n-k)!} \, a_k \, z^{n-1}, \end{aligned} \tag{6.4}$$

where $n = 1, 2, \cdots$

On the other hand, (6.4) can be written in matricial form as

$$\begin{bmatrix} \varphi_0/\varphi_1 \\ \varphi_1/\varphi_2 \\ \varphi_2/\varphi_3 \\ \varphi_3/\varphi_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1! & & & & \cdots \\ 2 & 2! & & & \cdots \\ 3 & 3\cdot 2 & 3! & & \cdots \\ 4 & 4\cdot 3 & 4\cdot 3\cdot 2 & 4! & \cdots \\ & & \vdots & & \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{bmatrix}.$$

Therefore, system (6.4) is solvable and

$$a_k = \sum_{l=1}^{k} (-1)^{k+l} \frac{1}{l!(k-l)!} \frac{\varphi_{l-1}}{\varphi_l}, \quad k = 1, 2, \cdots, \tag{6.5}$$

□

In order that the entire function φ satisfies the condition (6.3) in the definition of the GL derivative operator, we may assume that there exists $l_0 \in \mathbb{N}$ such that $\left|\frac{\varphi_{l-1}}{\varphi_l}\right| \le l^m, \quad m \in \mathbb{N}$ and $\frac{\varphi_{l-1}}{\varphi_l}$ is increasing for $l > l_0$. In this case, the coefficients a_k satisfy the following property.

Lemma 6.2. *If a_k is defined as in (6.5) for $k = 1, 2, \cdots$ and*

$$\left|\frac{\varphi_{l-1}}{\varphi_l}\right| \le l^m, \quad m \in \mathbb{N}, \quad l > l_0 \ (l_0 \in \mathbb{N})$$

and $\frac{\varphi_{l-1}}{\varphi_l}$ is increasing for $l > l_0$ $(l_0 \in \mathbb{N})$, then

$$\lim_{k\to\infty} |a_k| \to 0.$$

Proof. In fact, for $k \ge 2\min(l_0, m)$, we have

$$\begin{aligned}
|a_k| &\le \sum_{l=1}^{l_0} \frac{1}{l!(k-l)!} \left|\frac{\varphi_{l-1}}{\varphi_l}\right| + \sum_{l=l_0+1}^{k} \frac{1}{l!(k-l)!} l^m \\
&= \sum_{l=1}^{l_0} \frac{1}{l!(k-l)!} \left|\frac{\varphi_{l-1}}{\varphi_l}\right| + \frac{1}{k!} \sum_{l=l_0+1}^{k} \binom{k}{l} l^m \\
&= \sum_{l=1}^{l_0} \frac{1}{l!(k-l)!} \left(\left|\frac{\varphi_{l-1}}{\varphi_l}\right| - l^m\right) \\
&\quad + \frac{1}{k!} \left\{ 2^{k-m} \binom{k}{m} m! + 2 \sum_{l=1}^{m-1} (-1)^l \binom{k}{l} \frac{1}{2^l} \sum_{j=1}^{l} (-1)^j \binom{l}{j} j^m \right\} \\
&= \sum_{l=1}^{l_0} \frac{1}{l!(k-l)!} \left(\left|\frac{\varphi_{l-1}}{\varphi_l}\right| - l^m\right) \\
&\quad + \frac{2^{k-m}}{(k-m)!} + 2 \sum_{l=1}^{m-1} (-1)^l \frac{1}{l!(k-l)!} \frac{1}{2^l} \sum_{j=1}^{l} (-1)^j \binom{l}{j} j^m.
\end{aligned}$$

Since l_0 and m are fixed and $k \geq 2l_0$, we obtain that

$$\begin{aligned}
0 \leq \lim_{k\to\infty} \sum_{l=1}^{l_0} \frac{1}{l!(k-l)!} \left(\left| \frac{\varphi_{l-1}}{\varphi_l} \right| - l^m \right) &\leq \lim_{k\to\infty} \frac{1}{k!} \sum_{l=1}^{l_0} \binom{k}{l} \left(\left| \frac{\varphi_{l-1}}{\varphi_l} \right| - l^m \right) \\
&\leq \lim_{k\to\infty} \frac{1}{k!} \binom{k}{l_0} \underbrace{\left| \sum_{l=1}^{l_0} \left(\left| \frac{\varphi_{l-1}}{\varphi_l} \right| - l^m \right) \right|}_{constant\ independent\ of\ k} \\
&\to 0
\end{aligned}$$

so that $\lim_{k\to\infty} \sum_{l=1}^{l_0} \frac{1}{l!(k-l)!} \left(\left| \frac{\varphi_{l-1}}{\varphi_l} \right| - l^m \right) \to 0.$

On the other hand, since $k \geq 2m$, we can also obtain that

$$\begin{aligned}
\lim_{k\to\infty} \sum_{l=1}^{m-1} (-1)^l \frac{1}{l!(k-l)!} \frac{1}{2^l} \sum_{j=1}^{l} (-1)^j \binom{l}{j} j^m &= \lim_{k\to\infty} \frac{1}{k!} \sum_{l=1}^{m-1} \binom{k}{l} \frac{1}{2^l} \sum_{j=1}^{l} (-1)^{l+j} \binom{l}{j} j^m \\
&\leq \lim_{k\to\infty} \frac{1}{k!} \binom{k}{m-1} \underbrace{\sum_{l=1}^{m-1} \frac{1}{2^l} \max_{1<l<m-1} \left| \sum_{j=1}^{l} (-1)^{l+j} \binom{l}{j} j^m \right|}_{constant\ independent\ of\ k} \\
&\to 0.
\end{aligned}$$

As we have that $\lim_{k\to\infty} \frac{2^{k-m}}{(k-m)!} \to 0$, it follows that

$$\lim_{k\to\infty} |a_k| = 0.$$

□

Example 6.1. *If $\varphi(\lambda)$ is the Mittag-Leffler function (2.24) with $\mu = 1$ and $\rho = 1$ we have that*

$$\varphi(\lambda) = E_{1,1}(\lambda) = e^{\lambda} \tag{6.6}$$

and then $a_k = \begin{cases} 1, & k = 1; \\ 0, & k > 1 \end{cases}$ *meaning that*

$$D_\varphi f(z) = \sum_{k=1}^{\infty} a_k z^{k-1} \partial^k f(z) = \partial f(z).$$

In the next example, we establish the coefficients a_k for the case of $\varphi(z) = E_{\alpha,1}(z),\ \ 0 < \alpha < 1$, i.e. the Mittag-Leffler function.

Example 6.2. *The coefficients a_k in (6.5) constructed by the Mittag-Leffler function $E_{\alpha,1}$ are given by*

$$a_k = \sum_{l=1}^{k} (-1)^{k+l} \frac{1}{l!(k-l)!} \frac{\Gamma(\alpha l + 1)}{\Gamma(\alpha(l-1)+1)}, \quad k = 1, 2, \cdots .$$

By using Stirling's formula, we can prove that

$$\frac{\Gamma(\alpha l + 1)}{\Gamma(\alpha(l-1)+1)} \to (\alpha l)^{\alpha} \tag{6.7}$$

when $l \to \infty$ so that we have that $m = 1$. It is easy to obtain that the constant $l_0 = 1$.

Now, in order to define the fractional Laplace operator with respect to Gelfond-Leontiev operators of generalized differentiation (as we can see in the Definion 6.1), we will first establish a formula for the adjoint operator D_{φ}^{*} of the GL derivative operator in the next lemma.

Lemma 6.3. *The GL fractional derivative D_{φ} admits a (formal) adjoint D_{φ}^{*}.*

Proof. Using Lemma 6.1, we have that the GL derivative can be expressed as $D_{\varphi}f(z) = \sum_{k=1}^{\infty} a_k z^{k-1} \partial^k f(z)$, with a_k given by (6.5). Then we obtain that

$$D_{\varphi}^{*} g = \sum_{k=1}^{\infty} (-1)^k \overline{a}_k \overline{\partial}^k (\overline{z}^{k-1} g). \tag{6.8}$$

□

6.1.2 Explicit Formula for the Gelfond-Leontiev Laplace Operator

From Lemma 6.3, we now have the following expression for GL Laplace operator restricted to the real axis.

Theorem 6.4. *The GL Laplace operator is given by*

$$\Delta_{\varphi} = \sum_{m=2}^{\infty} b_m x^{m-2} \partial_x^m \tag{6.9}$$

where $b_m := \sum_{k=1}^{m-1} (-1)^k a_k \sum_{l=m-k}^{m} \begin{pmatrix} k \\ l+k-m \end{pmatrix} \frac{(k+l-2)!}{(m-2)!} a_l, \qquad m = 2, 3, \cdots .$

Proof. By Definition 6.1, we have

$$\begin{aligned} \Delta_{\varphi} = D_{\varphi}^{*} D_{\varphi} &= \sum_{k,l=1}^{\infty} (-\partial_x)^k \left(a_k a_l x^{k-1} x^{l-1} \partial_x^l \right) \\ &= \sum_{k,l=1}^{\infty} (-1)^k a_k a_l \partial_x^k \left(x^{k+l-2} \partial_x^l \right). \end{aligned} \tag{6.10}$$

Having in mind that

$$\partial_x^k \left(x^{k+l-2} \partial_x^l \right) = \sum_{j=0}^{k} \binom{k}{j} \frac{(k+l-2)!}{(k+l-2-j)!} \; x^{k+l-2-j} \partial_x^{l+k-j},$$

we obtain

$$\begin{aligned} \Delta_\varphi &= \sum_{k,l=1}^{\infty} (-1)^k a_k a_l \left[\sum_{j=0}^{k} \binom{k}{j} \frac{(k+l-2)!}{(k+l-2-j)!} \; x^{k+l-2-j} \partial_x^{l+k-j} \right] \\ &= \sum_{m=2}^{\infty} \left(\sum_{k=1}^{m-1} (-1)^k a_k \sum_{l=m-k}^{m} \binom{k}{l+k-m} \frac{(k+l-2)!}{(m-2)!} a_l \right) x^{m-2} \partial_x^m \\ &= \sum_{m=2}^{\infty} b_m x^{m-2} \partial_x^m, \end{aligned}$$

where $m = k + l - j$ and

$$b_m = \sum_{k=1}^{m-1} (-1)^k a_k \sum_{l=m-k}^{m} \binom{k}{l+k-m} \frac{(k+l-2)!}{(m-2)!} a_l \tag{6.11}$$

for $m = 2, 3, \ldots$. $\square$

Taking $i = l + k - m$, these above elements b_m can be rewritten as

$$b_m = \sum_{k=1}^{m-1} (-1)^k a_k \sum_{i=0}^{k} \binom{k}{i} (m-1)_i \; a_{i+m-k}, \qquad m = 2, 3, \ldots,$$

where $(m-1)_i = (m-1)m(m+1)\cdots(m-1+(i-1))$ is the Pochhammer symbol.

From Example 6.1, we can get the elements b_m in the classic case.

Example 6.3. *Since the coefficients a_k in Example 6.1 are given by* $a_k = \begin{cases} 1, & k = 1; \\ 0, & k > 1 \end{cases}$, *we have that*

$$b_m = \begin{cases} -1, & m = 2; \\ 0, & m > 2 \end{cases},$$

i.e we obtain

$$\Delta_\varphi = \sum_{m=2}^{\infty} b_m x^{m-2} \partial_x^m = -\partial_x^2.$$

Lemma 6.5. *Let a_k be the coefficients defined in 6.5 which satisfy*

$$\begin{cases} a_1 > 0, \\ \dfrac{a_k}{a_1} = (-1)^{k+1} \dfrac{1}{k^{[k/2]+q_k}}, \quad k = 2, 3, \cdots. \end{cases} \tag{6.12}$$

where $[k/2]$ is the integer part of $k/2$ and q_k is a real constant. If there exists $k_0 \in \mathbb{N}$ such that for $k > k_0$ we have $0 \leq q_k < [k/2]$ and $q_k \geq 2$ then

$$\lim_{m\to\infty} |b_m| = 0 \tag{6.13}$$

and

$$\lim_{k\to\infty} \left| \sum_{m=2}^{k} \frac{k!}{(k-m)!} b_m \right| = \infty. \tag{6.14}$$

Proof. First of all, one may observe that

$$|(m-1)_i| \leq c\, m^i, \tag{6.15}$$

where c is a real constant. Then, from (6.12) and (6.15) we obtain that

$$\begin{aligned} |b_m| &= \left| \sum_{k=1}^{m-1} (-1)^k a_k \sum_{i=0}^{k} \binom{k}{i} (m-1)_i \; a_{i+m-k} \right| \\ &\leq |c| \sum_{k=1}^{m-1} |a_k| \sum_{i=0}^{k} \binom{k}{i} m^i \; |a_{i+m-k}| \\ &= |c|(a_1)^2 \underbrace{\sum_{k=1}^{m-1} \frac{1}{k^{[k/2]+q_k}} \underbrace{\sum_{i=0}^{k} \binom{k}{i} \frac{m^i}{(i+m-k)^{[(i+m-k)/2]+q_{i+m-k}}}}_{(I)}}_{(II)} . \end{aligned}$$

In (I), we can note that

$$\sum_{i=0}^{k} \binom{k}{i} \frac{m^i}{(i+m-k)^{[(i+m-k)/2]+q_{i+m-k}}} = \binom{k}{0} \frac{m^0}{(m-k)^{[(m-k)/2]+q_{m-k}}}$$
$$+\binom{k}{1} \frac{m^1}{(1+m-k)^{[(1+m-k)/2]+q_{1+m-k}}} + \cdots + \binom{k}{k-1} \frac{m^{k-1}}{(m-1)^{[(m-1)/2]+q_{m-1}}} + \binom{k}{k} \frac{m^k}{m^{[m/2]+q_m}},$$

i.e., in this sum, we have quotients of powers in variable m and

$$\max_{0\leq i\leq k} \text{degree} \left(\binom{k}{i} \frac{m^i}{(i+m-k)^{[(i+m-k)/2]+q_{i+m-k}}} \right) = \text{degree} \left(\binom{k}{k} \frac{m^k}{m^{[m/2]+q_m}} \right). \tag{6.16}$$

From (6.16), it follows that (I) can be bounded as

$$\sum_{i=0}^{k} \binom{k}{i} \frac{m^i}{(i+m-k)^{[(i+m-k)/2]+q_{i+m-k}}} \quad \leq \quad (k+1) \binom{k}{k} \frac{m^k}{m^{[m/2]+q_m}}.$$

and thus, in (II) we obtain that

$$\sum_{k=1}^{m-1}\frac{1}{k^{[k/2]+q_k}}\sum_{i=0}^{k}\binom{k}{i}\frac{m^i}{(i+m-k)^{[(i+m-k)/2]+q_{i+m-k}}}\leq\sum_{k=1}^{m-1}\frac{1}{k^{[k/2]+q_k}}(k+1)\binom{k}{k}\frac{m^k}{m^{[m/2]+q_m}}$$
$$\leq(m-1)\frac{1}{(m-1)^{[(m-1)/2]+q_{m-1}}}m\frac{m^{m-1}}{m^{[m/2]+q_m}}=\frac{m^m}{m^{[m/2]+q_m}(m-1)^{[(m-1)/2]-1+q_{m-1}}}$$
$$\leq\frac{m^m}{(m-1)^{[m/2]+q_m}(m-1)^{[(m-1)/2]-1+q_{m-1}}}=\frac{m^m}{(m-1)^{[m/2]+[(m-1)/2]-1+q_m+q_{m-1}}}$$
$$=\frac{m^m}{(m-1)^{m-2+q_m+q_{m-1}}}\leq\frac{m^m}{(m-1)^{m+2}}=\left(\frac{m}{m-1}\right)^m\frac{1}{(m-1)^2}$$

where $[m/2]+[(m-1)/2]=m-1$ and by using the fact that $q_m+q_{m-1}\geq 4$.

Therefore, we have

$$\begin{aligned}\lim_{m\to\infty}|b_m| &= \lim_{m\to\infty}|c|(a_1)^2\sum_{k=1}^{m-1}\frac{1}{k^{[k/2]+q_k}}\sum_{i=0}^{k}\binom{k}{i}\frac{m^i}{(i+m-k)^{[(i+m-k)/2]+q_{i+m-k}}}\\ & |c|(a_1)^2\lim_{m\to\infty}\left(\frac{m}{m-1}\right)^m\frac{1}{(m-1)^2}\\ &= |c|(a_1)^2 e\lim_{m\to\infty}\frac{1}{(m-1)^2}\\ &= 0,\end{aligned}$$

so that (6.13) is proved.

In the second part of this lemma, by using (6.12) and the fact that $(m-1)_i=p_i(m)$, i.e it is a polynomial of degree i in the variable m, we obtain that

$$\begin{aligned}b_m &= \sum_{k=1}^{m-1}(-1)^k a_k\sum_{i=0}^{k}\binom{k}{i}(m-1)_i\ a_{i+m-k}\\ &= a_1^2\sum_{k=1}^{m-1}\frac{1}{k^{[k/2]+q_k}}\sum_{i=0}^{k}\binom{k}{i}p_i(m)\ (-1)^{i+m-k}\frac{1}{(i+m-k)^{[(i+m-k)/2]+q_{i+m-k}}}.\end{aligned}$$

From the fact that b_m satisfies

$$\frac{s!}{(s-m)!}b_m=\frac{s!}{(s-m)!}a_1^2\sum_{k=1}^{m-1}\frac{1}{k^{[k/2]+q_k}}\sum_{i=0}^{k}\binom{k}{i}p_i(m)\ (-1)^{i+m-k}\frac{1}{(i+m-k)^{[(i+m-k)/2]+q_{i+m-k}}}$$
$$>a_1^2\sum_{k=1}^{m-1}\frac{1}{k^{[k/2]+q_k}}\frac{s!}{(s-m)!}\min\left(\binom{k}{k}\frac{m^k}{m^{[m/2]+q_m}}-\binom{k}{k-1}\frac{m^{k-1}}{(m-1)^{[(m-1)/2]+q_{m-1}}}\right)$$
$$=a_1^2\sum_{k=1}^{m-1}\frac{1}{k^{[k/2]+q_k}}\frac{s!}{(s-m)!}\min\left(\frac{m^{k-1}}{m^{[(m-1)/2]}}\left[1-\left(\frac{m}{m-1}\right)^{[(m-1)/2]}\frac{m^{q_m}}{(m-1)^{q_{m-1}}}\right]\right)$$

we get

$$\lim_{s,m\to\infty} \frac{s!}{(s-m)!} b_m \quad > \quad c_0,$$

where c_0 is real constant such that $c_0 \neq 0$. Consequently, we have

$$\lim_{k\to\infty} \left| \sum_{m=2}^{k} \frac{k!}{(k-m)!} b_m \right| = \infty.$$

□

Remark 6.6. *Condition (6.12) in Lemma (6.5) may look strange at first sight but we can check numerically that in the case of the Mittag-Leffler function $E_{\alpha,1}$, with $0 < \alpha < 1$, this condition is satisfied for the first 50 terms which indicates that imposing this condition looks natural. For more details see Appendix A.*

Lemma 6.7. *The fractional Laplace operator associated to the GL fractional derivative D_φ is a self-adjoint operator.*

Proof. $\Delta_\varphi^* = (D_\varphi^* D_\varphi)^* = D_\varphi^* D_\varphi = \Delta_\varphi$. □

Definition 6.2. *The Fourier symbol for the GL fractional Laplacian Δ_φ is given by*

$$\sigma_\varphi(x,\xi) = \sum_{k=2}^{\infty} b_m x^{m-2} \xi^m. \tag{6.17}$$

It is known that the operator Δ_φ is elliptic if there exist an $\epsilon > 0$ and a constant $C_\epsilon > 0$ such that

$$|\sigma_\varphi(x,\xi)| \geq \epsilon(1 + |\xi|), \quad |\xi| \geq C_\epsilon. \tag{6.18}$$

Example 6.4. *In the case of $\varphi(\lambda) = E_{\alpha,1}(\lambda)$ being the Mittag-Leffler function with $0 < \alpha < 1$ we have the Δ_φ is elliptic and self-adjoint.*

We now look into eigenfunctions of the GL fractional Laplace operator (6.9). Since the GL derivatives act on analytic functions, we seek $\psi \in \mathcal{A}$ such that

$$\psi(x) = \sum_{n=0}^{\infty} c_n x^n. \tag{6.19}$$

Here, we have

$$\begin{aligned}
\Delta_\varphi \psi(x) &= \sum_{m=2}^{\infty} b_m x^{m-2} D^m \sum_{n=0}^{\infty} c_n x^n \\
&= \sum_{n=2}^{\infty}\sum_{m=2}^{n} b_m x^{m-2} c_n \frac{n!}{(n-m)!} x^{n-m} \\
&= \sum_{n=2}^{\infty}\sum_{m=2}^{n} b_m c_n \frac{n!}{(n-m)!} x^{n-2} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=2}^{n+2} \frac{(n+2)!}{(n+2-m)!} b_m c_{n+2} \right) x^n
\end{aligned}$$

so the generalized eigenfunctions arise from solving the system

$$c_n = \sum_{m=2}^{n+2} \frac{(n+2)!}{(n+2-m)!} b_m c_{n+2}, \qquad n = 0, 1, 2, \cdots . \tag{6.20}$$

From (6.20) we can obtain that the constants of $\psi(x)$ are given by

$$c_{2n} = \frac{1}{\prod_{k=1}^{n} \sum_{m=2}^{2k} \frac{(2k)!}{(2k-m)!} b_m} c_0, \qquad c_{2n+1} = \frac{1}{\prod_{k=1}^{n} \sum_{m=2}^{2k+1} \frac{(2k+1)!}{(2k+1-m)!} b_m} c_1$$

$n = 1, 2, \cdots$ and then, $\psi(x)$ has the following form:

$$\psi(x) = c_0 + c_1 x + c_0 \sum_{n=1}^{\infty} \frac{1}{\prod_{k=1}^{n} \sum_{m=2}^{2k} \frac{(2k)!}{(2k-m)!} b_m} x^{2n} + c_1 \sum_{n=1}^{\infty} \frac{1}{\prod_{k=1}^{n} \sum_{m=2}^{2k+1} \frac{(2k+1)!}{(2k+1-m)!} b_m} x^{2n+1} \tag{6.21}$$

which is a entire function since we have (6.14). Indeed, we can obtain

$$\lim_{n\to\infty} \left| \frac{c_{2n}}{c_{2n+2}} \right| = \lim_{n\to\infty} \left| \sum_{m=2}^{2n+2} \frac{(2n+2)!}{(2n+2-m)!} b_m \right| = \infty$$

and similarly,

$$\lim_{n\to\infty} \left| \frac{c_{2n+1}}{c_{2n+3}} \right| = \infty.$$

From the Example (6.3), we can obtain the constants c_{2n} and c_{2n+1} $(n = 1, 2, \cdots)$ in the classic case which are given by

$$\begin{aligned}
c_{2n} &= \frac{1}{\prod_{k=1}^{n}\sum_{m=2}^{2k}\frac{(2k)!}{(2k-m)!}b_m}c_0\\
&= \frac{1}{\prod_{k=1}^{n}\frac{(2k)!}{(2k-2)!}b_2}c_0\\
&= \frac{(-1)^n}{\prod_{k=1}^{n}(2k)(2k-1)}c_0\\
&= \frac{(-1)^n}{(2n)!}c_0
\end{aligned}$$

and

$$\begin{aligned}
c_{2n+1} &= \frac{1}{\prod_{k=1}^{n}\sum_{m=2}^{2k+1}\frac{(2k+1)!}{(2k+1-m)!}b_m}c_1\\
&= \frac{1}{\prod_{k=1}^{n}\frac{(2k+1)!}{(2k-1)!}b_2}c_1\\
&= \frac{(-1)^n}{\prod_{k=1}^{n}(2k+1)(2k)}c_1\\
&= \frac{(-1)^n}{(2n+1)!}c_1,
\end{aligned}$$

so that, if $c_0 = c_1 = 1$ the $\psi(x)$ has the form:

$$\psi(x) = 1 + x + \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} + \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} = e^{-x}. \tag{6.22}$$

Thus, it follows the next theorem.

Theorem 6.8. *The generalized eigenfunctions of the GL fractional Laplace operator defined by (6.9) are the analytic functions* $\psi(x) = \sum_{n=0}^{\infty} c_n x^n$ *where the coefficients are given in (6.21).*

6.2 Fourier Fractional Transform

The classic Fourier transform is defined as the integration of an L_1-function against the eigenfunctions of the Laplacian. In order to follow this idea for the definition of our integral transform we need generalized eigenfunctions which are bounded over $\mathbb{R}$. This leads to the system of eigenfunctions $\{\psi_\lambda = \psi(i\lambda x), \lambda \in \mathbb{R}\}$ under the condition that $c_n \geq 0$ for all n. In the case of the Mittag-Leffler function $\varphi(\lambda) = E_{\frac{1}{\rho},\mu}(\lambda)$, it can be verified as we can see in the following example.

Example 6.5. *Let $\varphi(\lambda) = E_{\frac{1}{\rho},\mu}(\lambda)$ be the Mittag-Leffler function. Then we have that*

$$\varphi(ix) = \sum_{k=0}^{\infty} \frac{i^k x^k}{\Gamma\left(\mu + \frac{k}{\rho}\right)} \tag{6.23}$$

converges for all $x \in (-\infty, \infty)$ and $\mu, \frac{1}{\rho} \in \mathbb{R}_+$. Moreover, we have $\phi(x) = \varphi(ix) \in L_2(\mathbb{R}, dx)$. In fact, by using the Stirling's formula

$$\lim_{x\to\infty} \frac{\Gamma(x)}{\sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x}} = 1. \tag{6.24}$$

and taking $x \in \mathbb{R}_+$ and $z \in \mathbb{C}$ we obtain that

$$\lim_{x\to\infty} \frac{\Gamma(x+z)}{\Gamma(x)x^z} = \lim_{x\to\infty} \frac{\Gamma(x+z)}{\sqrt{2\pi}(x+z)^{x+z-\frac{1}{2}} e^{-(x+z)}} \left(\frac{\Gamma(x)}{\sqrt{2\pi} e^{-x} x^{x-\frac{1}{2}}} \frac{x^z x^{x-\frac{1}{2}}}{(x+z)^{x+z-\frac{1}{2}} e^{-z}} \right)^{-1}$$
$$= \lim_{x\to\infty} \frac{(x+z)^{x+z-\frac{1}{2}} e^{-z}}{x^z x^{x-\frac{1}{2}}} = e^{-z} \lim_{x\to\infty} (1 + \frac{z}{x})^{x+z-\frac{1}{2}} = e^{-z} e^{z} \lim_{x\to\infty} (1+\frac{z}{x})^{z-\frac{1}{2}} = 1,$$

that is $\lim_{x\to\infty} \frac{\Gamma(x+z)}{\Gamma(x)x^z} = 1$.

Now, from

$$E_{\frac{1}{\rho},\mu}(ix) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{\Gamma\left(\mu + \frac{2k}{\rho}\right)} + i \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{\Gamma\left(\mu + \frac{2k+1}{\rho}\right)}$$

and since the fact that $\Gamma(t)$ is monotone increasing on $(\frac{3}{2}, +\infty)$ and, consequently, $\left|\frac{1}{\Gamma(\mu + \frac{k}{\rho})}\right| \to 0$ for $k \to \infty$ we get

$$\begin{aligned} \lim_{k\to\infty} \left| \frac{\frac{x^{2(k+1)}}{\Gamma\left(\mu + \frac{2(k+1)}{\rho}\right)}}{\frac{x^{2k}}{\Gamma\left(\mu+\frac{2k}{\rho}\right)}} \right| &= |x|^2 \lim_{k\to\infty} \frac{\Gamma\left(\mu + \frac{2k}{\rho}\right)}{\Gamma\left(\mu + \frac{2(k+1)}{\rho}\right)} \\ &= 0 \end{aligned}$$

and in the same way,

$$\lim_{k\to\infty} \left| \frac{\frac{x^{2(k+1)+1}}{\Gamma\left(\mu + \frac{2(k+1)+1}{\rho}\right)}}{\frac{x^{2k+1}}{\Gamma\left(\mu+\frac{2k+1}{\rho}\right)}} \right| = 0.$$

Then, we have $\varphi(ix)$ converges for all $x \in (-\infty, \infty)$.

Finally, Using (6.23), we obtain that $\sup|\varphi(ix)|^2 = C < \infty$ *and consequently,*

$$\int_{-\infty}^{\infty} |\varphi(ix)|^2 dx < \infty,$$

i.e., $\varphi(ix) \in L_2(\mathbb{R}, dx)$.

Now we consider eigenfunctions $\psi_\lambda = \psi(i\lambda x)$ which are bounded over $\mathbb{R}$.

Lemma 6.9. *The family of eigenfunctions of* Δ_φ, $\{\psi_\lambda, \lambda \in \mathbb{R} : \Delta_\varphi \psi_\lambda = -\lambda^2 \psi_\lambda\}$ *is an orthogonal basis with respect to* $L_2(\mathbb{R}, d\mu(x))$, *where* μ *denotes the spectral measure associated to* Δ_φ.

Let us remark that the inner product with respect to the spectral measure μ can also be written as

$$\int_{\mathbb{R}} f(x)\overline{g(x)}d\mu = \int_{\mathbb{R}} \Delta_\varphi f(x)\overline{g(x)}dx = \int_{\mathbb{R}} D_\varphi f(x)\overline{D_\varphi g(x)}dx$$

since under our established conditions the GL Laplace operator is a symmetric and positive operator. This allows us to define our fractional Fourier-like transform.

Definition 6.3. *Given an entire function* φ *with order* $\rho > 0$ *and degree* $\sigma > 0$ *we define the Fourier-like transform associated to the GL derivative* D_φ *as*

$$\mathcal{F}_\varphi f(\xi) := \langle f, \overline{\psi(i\xi\cdot)}\rangle = \int_{-\infty}^{\infty} f(x)\overline{\psi(i\xi x)}dx, \tag{6.25}$$

where $f \in L_1(\mathbb{R}, d\mu(x))$. *The definition can be extended to* $L_2(\mathbb{R}, d\mu(x))$ *in the usual way.*

For the eigenfunctions $\psi_\lambda(x) = e^{i\lambda x}$ (see (6.22)), we can obtain the classic definition of Fourier transform, which is given by

$$\mathcal{F} f(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx = \int_{-\infty}^{\infty} f(x)\overline{e^{i\xi x}}dx,$$

The next theorem give us some properties of the Fourier-like transform associated to the GL derivative.

Theorem 6.10. *The Fourier like transform associated to the GL derivative operator satisfies the following properties:*

(a) $\mathcal{F}_\varphi f$ *exists if* $f \in L_1(\mathbb{R}, dx)$;

(b) *Inversion formula: Let* $f \in L_1(\mathbb{R}, dx) \cap L_2(\mathbb{R}, dx)$ *and* $F = \mathcal{F}_\varphi f$ *then* $(\mathcal{F}_\varphi^{-1} F)(x) = \int_{\mathbb{R}} F(\xi)\psi(i\xi x)d\mu(\xi)$;

(c) *Linearity:* $\mathcal{F}_\varphi(f+g)(\xi) = \mathcal{F}_\varphi f(\xi) + \mathcal{F}_\varphi g(\xi)$

(d) *Plancherel formula: Let* $f, g \in L_1(\mathbb{R}, dx) \cap L_2(\mathbb{R}, dx)$ *then* $\langle \mathcal{F}_\varphi f, \mathcal{F}_\varphi g\rangle_\mu = \langle f, g\rangle$*;*

(e) $\mathcal{F}_\varphi : L_2(\mathbb{R}, dx) \to L_2(\mathbb{R}, d\mu(x))$ *is an isometry.*

Hereby $\langle \cdot, \cdot\rangle_\mu$ *denotes the inner product with respect to the spectral measure* μ*.*

Proof. Observing the fact that the generalized eigenfunctions ψ_λ are bounded over $\mathbb{R}$, (a) can be easily verified, i.e.

$$|\mathcal{F}_\varphi f(\xi)| = |\int_{-\infty}^{\infty} f(x)\overline{\psi(i\xi x)}dx| \leq \sup_{x\in\mathbb{R}} |\overline{\psi(i\xi x)}| \int_{-\infty}^{\infty} |f(x)|dx \leq C\|f\|_{L_1}.$$

For the inversion formula we note that by using $F(\xi) = \mathcal{F}_\varphi f(\xi)$ and (6.25), we obtain that first that $F \in L_2(\mathbb{R}, d\mu)$ since

$$\begin{aligned}
||\mathcal{F}_\varphi f(\xi)||^2_{L_2} &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x)\overline{\psi(i\xi x)}dx \int_{-\infty}^{\infty} \overline{f(y)}\psi(i\xi y)dy\right) d\mu(\xi) \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x)\overline{f(y)} \left(\int_{-\infty}^{\infty} \overline{\psi(i\xi x)}\psi(i\xi y)d\mu(\xi)\right) dx\right) dy \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x)\overline{f(y)}\delta(y-x)dy\right) dx \\
&= \int_{-\infty}^{\infty} |f(x)|^2 dx \\
&= ||f||^2_{L_2} < \infty,
\end{aligned}$$

since ψ satisfies Lemma 6.9 and $\int_{-\infty}^{\infty} \overline{\psi(i\xi y)}\psi(i\xi x)d\mu(\xi) = \langle \psi(i\cdot y), \psi(-i\cdot x)\rangle_\mu = \delta(y-x)$. This formula is also known as Parseval formula and also proves property (e).

Now, we have

$$\begin{aligned}
(\mathcal{F}_\varphi^{-1} F)(x) &= \int_{\mathbb{R}} F(\xi)\psi(i\xi x)d\mu(\xi) \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y)\overline{\psi(i\xi y)}dy\right) \psi(i\xi x)d\mu(\xi) \\
&= \int_{-\infty}^{\infty} f(y) \left(\int_{-\infty}^{\infty} \overline{\psi(i\xi y)}\psi(i\xi x)d\mu(\xi)\right) dy \\
&= \int_{-\infty}^{\infty} f(y)\delta(y-x)dy \\
&= f(x).
\end{aligned}$$

Thus, it is proved (b).

Linearity can be easily checked. Then, using the fact that

$$f(y) = \int_{-\infty}^{\infty} \mathcal{F}_\varphi f(\xi)\psi(i\xi y)d\mu(\xi) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x)\overline{\psi(i\xi x)}dx\right) \psi(i\xi y)d\mu(\xi),$$

we have

$$\begin{aligned}\langle f,g\rangle &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\overline{\psi(i\xi x)}\,dx\,\psi(i\xi y)\,d\mu(\xi)\right)g(y)dy\\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x)\overline{\psi(i\xi x)}dx\right)\overline{\left(\int_{-\infty}^{\infty} g(y)\overline{\psi(i\xi y)}dy\right)}d\mu(\xi)\\ &= \langle\mathcal{F}_\varphi f,\mathcal{F}_\varphi g\rangle\end{aligned}$$

so that we have (d). □

Let us remark that we used the classic approach to the Fourier transform of using functions $L_1(\mathbb{R},dx)\cap L_2(\mathbb{R},dx)$. Our approach would also allow us to study the Fourier transform for $L_2(\mathbb{R},dx)$ via the Gelfand triple $\mathcal{S}\subset L_2(\mathbb{R},dx)\subset\mathcal{S}'$ where $\mathcal{S}$ denotes the space of rapidly decaying functions and $\mathcal{S}'$ the space of temperate distributions. Therefore, from now on we will consider the fractional Fourier-like transform for $L_2(\mathbb{R},dx)$.

6.3 Convolution

One of the advantages of the classic Fourier transform is that it maps the convolution of two functions into the pointwise multiplication of their transforms. While this is not anymore true for other integral transforms we can still consider the pointwise multiplication of the transforms of two functions. This allows us to define a convolution in this case, i.e. if $f,g\in L_2(\mathbb{R},dx)$ with fractional Fourier-like transforms associated to the GL derivative $\mathcal{F}_\varphi f$ and $\mathcal{F}_\varphi g$ respectively, then the convolution is given by inverse transform applied to the product of the fractional Fourier-like transforms $\mathcal{F}_\varphi f$ and $\mathcal{F}_\varphi g$. This leads to the following definition.

Definition 6.4. *Let f,g be functions in $L_2(\mathbb{R},dx)$ and let $\mathcal{F}_\varphi f,\mathcal{F}_\varphi g$ be their fractional Fourier-like transforms, respectively. The convolution of f with g is the function $f*g$ given by*

$$(f*g)(x)=\mathcal{F}_\varphi^{-1}\left(\mathcal{F}_\varphi f\mathcal{F}_\varphi g\right)(x) \tag{6.26}$$

Let us first show that this definition makes sense. From the previous section we have that if f,g belong to $L_2(\mathbb{R},dx)$ then their transforms $\mathcal{F}_\varphi f,\mathcal{F}_\varphi g$ belong to $L_2(\mathbb{R},d\mu(\xi))$. From the elementary in inequality $2|\mathcal{F}_\varphi f\mathcal{F}_\varphi g|\le 2|\mathcal{F}_\varphi f||\mathcal{F}_\varphi g|\le|\mathcal{F}_\varphi f|^2+|\mathcal{F}_\varphi g|^2$ we can deduce that the product of $\mathcal{F}_\varphi f$ and $\mathcal{F}_\varphi g$ belongs to $L_1(\mathbb{R},d\mu)$, i.e.

$$\int_{-\infty}^{\infty}|\mathcal{F}_\varphi f(\xi)\mathcal{F}_\varphi g(\xi)|d\mu(\xi)\le\frac{1}{2}\left(\int_{-\infty}^{\infty}|\mathcal{F}_\varphi f(\xi)|^2d\mu(\xi)+\int_{-\infty}^{\infty}|\mathcal{F}_\varphi g(\xi)|^2d\mu(\xi)\right)<\infty$$

so that we can apply the inverse fractional Fourier-like transform. Consequently, the term $(f*g)(x)$ makes sense for all $x\in\mathbb{R}$.

For this convolution we have the following properties.

Theorem 6.11. *For functions $f, g, h \in L_2(\mathbb{R}, dx)$ it holds:*

1. *Commutativity:* $f * g = g * f$*;*
2. *Associativity:* $((f * g) * h) = (f * (g * h))$*;*
3. *Distributivity:* $[f * (g + h)] = (f * g) + (f * h)$
4. *Differentiation:* $(D_\varphi)(f * g) = \mathcal{F}_\varphi^{-1}[\mathcal{F}_\varphi(D_\varphi f)\mathcal{F}_\varphi g] = \mathcal{F}_\varphi^{-1}[\mathcal{F}_\varphi(f)\mathcal{F}_\varphi(D_\varphi g)]$.

Proof. Property 1 is easy to check. By using Definition 6.26 Property 2 is verified by the following calculation

$$\begin{aligned}
((f * g) * h)(x) &= \mathcal{F}_\varphi^{-1}[(\mathcal{F}_\varphi(f * g))(\mathcal{F}_\varphi h)] \\
&= \mathcal{F}_\varphi^{-1}\left\{\left[\mathcal{F}_\varphi \mathcal{F}_\varphi^{-1}(\mathcal{F}_\varphi f \mathcal{F}_\varphi g)\right](\mathcal{F}_\varphi h)\right\} \\
&= \mathcal{F}_\varphi^{-1}[(\mathcal{F}_\varphi f)(\mathcal{F}_\varphi g)(\mathcal{F}_\varphi h)] \\
&= \mathcal{F}_\varphi^{-1}\left\{(\mathcal{F}_\varphi f)\left[\mathcal{F}_\varphi \mathcal{F}_\varphi^{-1}(\mathcal{F}_\varphi g \mathcal{F}_\varphi h)\right]\right\} \\
&= \mathcal{F}_\varphi^{-1}[(\mathcal{F}_\varphi f)(\mathcal{F}_\varphi(g * h))] \\
&= (f * (g * h))(x).
\end{aligned}$$

Property 3 is proved by using the linearity property of the fractional Fourier-like transform and its inverse, as we can see

$$\begin{aligned}
(f * (g + h))(x) &= \\
&= \mathcal{F}_\varphi^{-1}[(\mathcal{F}_\varphi f)(\mathcal{F}_\varphi(g + h))] \\
&= \mathcal{F}_\varphi^{-1}[(\mathcal{F}_\varphi f)(\mathcal{F}_\varphi g) + (\mathcal{F}_\varphi f)(\mathcal{F}_\varphi h)] \\
&= \mathcal{F}_\varphi^{-1}[(\mathcal{F}_\varphi(f))(\mathcal{F}_\varphi g)] + \mathcal{F}_\varphi^{-1}[(\mathcal{F}_\varphi f)(\mathcal{F}_\varphi h)] \\
&= (f * g)(x) + (f * h)(x).
\end{aligned}$$

Now, taking the derivative D_φ of $f * g$ with respect to x we obtain

$$\begin{aligned}
(D_\varphi)_x(f * g)(x) &= (D_\varphi)_x \mathcal{F}_\varphi^{-1}(\mathcal{F}_\varphi f \mathcal{F}_\varphi g)(x) \\
&= \mathcal{F}_\varphi^{-1}(i\xi(\mathcal{F}_\varphi f(\xi)\mathcal{F}_\varphi g(\xi)))(x) \\
&= \mathcal{F}_\varphi^{-1}([\mathcal{F}_\varphi((D_\varphi)_x f)]\mathcal{F}_\varphi g)(x).
\end{aligned}$$

By invoking the commutativity of convolution (see Property 1), we get

$$(D_\varphi)_x(f * g)(x) = \mathcal{F}_\varphi^{-1}(\mathcal{F}_\varphi f[\mathcal{F}_\varphi((D_\varphi)_x g)])(x)$$

so that Property 4 is proved. $\square$

Remark 6.12. *For the convolution we can formally define a neutral element g as*

$$g(x) = (\mathcal{F}_{\varphi}^{-1}1)(x) = \int_{-\infty}^{\infty} \psi(i\xi x)d\mu(\xi),$$

but we need to keep in mind that the constant function 1 does not belong to $L_2(\mathbb{R}, d\mu)$ and, therefore, g cannot be a classic function. g only makes sense as a temperate distribution, i.e. $g \in \mathcal{S}'$.

Chapter 7

Conclusion and Outlook

This dissertation presents the fundamentals of a fractional function theory in higher dimensions based on the Gelfond-Leontiev (GL) operators of generalized differentiation.

This theory was constructed in the context of classic Clifford algebras as well as of generalized Clifford algebras, in particular ternary Clifford algebras. Some preliminary ideas about a function theory in higher dimensions and the definition of the GL operators of generalized differentiation were presented in Chapters 1 and 2, respectively. Connections between the definitions of GL operators of generalized integration and Riemann-Liouville integral operator were presented in the Chapter 2.

The fractional function theory for classic Clifford algebras is given in the first section of the Chapter 4 where the Dirac operator based on GL fractional derivatives still allowed for a Fischer decomposition and the explicit construction of the basis of fractional homogeneous monogenic polynomials acting on a ground state. The reproducing kernel Hilbert space of monogenic functions are constructed in the Chapter 5, with the particular cases of the Drury-Arveson space and de Branges-Rovnyak spaces associated to Schur multipliers in mind. Moreover, based on the basic monogenic powers and Fueter series we studied Gleason's problem in this context and presented its link with Leibenson's shift operators.

The ternary algebras were presented in Chapter 3 having its corresponding function theory described in the second section of Chapter 4. In ternary algebras the corresponding Dirac operator provides a cubic factorization of the Laplacian. The construction of this theory is based on a Fischer pair and by introducing an inner product onto the space of homogeneous polynomials with values in the generalized Clifford algebra we constructed the Fischer decomposition. Thereby we used a computer algebra system to compute the coefficients of the monogenic homogeneous polynomials that form the basis of the space of fractional homogeneous monogenic polynomials which arise in this case. The construction of the reproducing kernel Hilbert spaces of monogenic functions was presented in Chapter 5 for the particular cases of the Drury-Arveson space and de Branges-Rovnyak spaces associated to Schur multipliers. By using the basic monogenic powers and Fueter series we studied the Gleason's problem in this context and again presented its link with Leibenson's shift operators.

In the Chapter 6, by using the fact that the Gelfond-Leontiev derivative operator can be written as a series of the standard derivatives we obtain an explicit formula for the Gelfond-Leontiev Laplace operator and calculated its generalized eigenfunctions. The fractional Fourier-like transform associated to the Gelfond-Leontiev derivative operator was defined and a definition of the corresponding convolution given.

The major contribution of this thesis to the development of fractional function theory is due to the fact that in the ternary Clifford algebras the corresponding Dirac operator provides a cubic factorization of the Laplacian and we provide the fundamentals for its associated function theory.

In conclusion, the key in this approach is the Gelfond-Leontiev operator of generalized differentiation which is the generalization of the standard derivative operator and it was used to defined the Dirac operator so that we can construct all the facts in this theory.

Appendix A

Computational calculations

If $\alpha = \frac{2}{3}$ then the elements a_k for the Mittag-Leffler function $\varphi(\lambda) = E_{\frac{2}{3},1}(\lambda)$ is given by

$$a_k = \sum_{l=1}^{k} (-1)^{k+l} \frac{1}{l!(k-l)!} \frac{\Gamma(\frac{2}{3}l+1)}{\Gamma(\frac{2}{3}(l-1)+1)}, \quad k = 1, 2, \cdots .$$

The first 50 terms of a_k:

$a_1 = 0,902745$	$a_2 = -0,243291$	$a_3 = 0,0718795$	$a_4 = -0,0171043$
$a_5 = 0,00333265$	$a_6 = -0,000546695$	$a_7 = 0,0000772815$	$a_8 = -9,5888 \times 10^{-6}$
$a_9 = 1,05961 \times 10^{-6}$	$a_{10} = -1,05521 \times 10^{-7}$	$a_{11} = 9,56149 \times 10^{-9}$	$a_{12} = -7,94707 \times 10^{-10}$
$a_{13} = 6,10007 \times 10^{-11}$	$a_{14} = -4,34947 \times 10^{-12}$	$a_{15} = 2,89533 \times 10^{-13}$	$a_{16} = -1,80729 \times 10^{-14}$
$a_{17} = 1,06196 \times 10^{-15}$	$a_{18} = -5,8942 \times 10^{-17}$	$a_{19} = 3,09966 \times 10^{-18}$	$a_{20} = -1,54871 \times 10^{-19}$
$a_{21} = 7,37007 \times 10^{-21}$	$a_{22} = -3,34812 \times 10^{-22}$	$a_{23} = 1,45496 \times 10^{-23}$	$a_{24} = -6,05956 \times 10^{-25}$
$a_{25} = 2,42282 \times 10^{-26}$	$a_{26} = -9,31504 \times 10^{-28}$	$a_{27} = 3,44883 \times 10^{-29}$	$a_{28} = -1,23134 \times 10^{-30}$
$a_{29} = 4,24478 \times 10^{-32}$	$a_{30} = -1,41455 \times 10^{-33}$	$a_{31} = 4,56197 \times 10^{-35}$	$a_{32} = -1,4253 \times 10^{-36}$
$a_{33} = 4,31817 \times 10^{-38}$	$a_{34} = -1,26978 \times 10^{-39}$	$a_{35} = 3,62726 \times 10^{-41}$	$a_{36} = -1,00729 \times 10^{-42}$
$a_{37} = 2,72131 \times 10^{-44}$	$a_{38} = -7,15766 \times 10^{-46}$	$a_{39} = 1,83405 \times 10^{-47}$	$a_{40} = -4,57544 \times 10^{-49}$
$a_{41} = 1,11165 \times 10^{-50}$	$a_{42} = -2,63949 \times 10^{-52}$	$a_{43} = 6,07669 \times 10^{-54}$	$a_{44} = -1,38599 \times 10^{-55}$
$a_{45} = 3,26584 \times 10^{-57}$	$a_{46} = -7,46762 \times 10^{-59}$	$a_{47} = 1,98359 \times 10^{-60}$	$a_{48} = -5,46945 \times 10^{-62}$
$a_{49} = 2,08902 \times 10^{-63}$	$a_{50} = -7,83384 \times 10^{-65}$		

We remind that the condition (6.12), for constants a_k, in the Lemma 6.5 is given as

$$\begin{cases} a_1 > 0, & ; \\ \dfrac{a_k}{a_1} = (-1)^{k+1} \dfrac{1}{k^{[k/2]+q_k}}, & k = 2, 3, \cdots . \end{cases}$$

where

- $[k/2]$ is the integral part of $k/2$;
- q_k is a real constant and there exists $k_0 \in \mathbb{N}$ such that if $k \geq k_0$ then $0 \leq q_k < [k/2]$ and $q_k \geq 2$.

We already have

- $a_1 > 0$
- the constants a_k are alternating

From

$$\left|\frac{a_k}{a_1}\right| = \left|(-1)^{k+1}\frac{1}{k^{[k/2]+q_k}}\right| \Leftrightarrow q_k = \log_k\left(\frac{a_1}{k^{[k/2]|a_k|}}\right)$$

we obtain the first 50 constants q_k as:

$q_1 = 0$	$q_2 = 0,891639$	$q_3 = 1,30331$	$q_4 = 0,860945$	$q_5 = 1,48052$
$q_6 = 1,13521$	$q_7 = 1,81304$	$q_8 = 1,50754$	$q_9 = 2,21479$	$q_{10} = 1,93223$
$q_{11} = 2,65805$	$q_{12} = 2,39095$	$q_{13} = 3,12994$	$q_4 = 2,87423$	$q_{15} = 3,62321$
$q_{16} = 3,37639$	$q_{17} = 4,13334$	$q_{18} = 3,89373$	$q_{19} = 4,65725$	$q_{20} = 4,42365$
$q_{21} = 5,19271$	$q_{22} = 4,96424$	$q_{23} = 5,73808$	$q_{24} = 5,51408$	$q_{25} = 6,29209$
$q_{26} = 6,07201$	$q_{27} = 6,85372$	$q_{28} = 6,63713$	$q_{29} = 7,42215$	$q_{30} = 7,2087$
$q_{31} = 7,99671$	$q_{32} = 7,78611$	$q_{33} = 8,57683$	$q_{34} = 8,36884$	$q_{35} = 9,16205$
$q_{36} = 8,95646$	$q_{37} = 9,75203$	$q_{38} = 9,54872$	$q_{39} = 10,3465$	$q_{40} = 10,1456$
$q_{41} = 10,9462$	$q_{42} = 1,7475$	$q_{43} = 11,5515$	$q_{44} = 11,3528$	$q_{45} = 12,1405$
$q_{46} = 11,9314$	$q_{47} = 12,6786$	$q_{48} = 12,4122$	$q_{49} = 13,0582$	$q_{50} = 12,7061$

In order to see that $q_k < [k/2]$, we now compute $s_k = q_k - [k/2]$:

$s_1 = 0$	$s_2 = -0,108361$	$s_3 = 0,303314$	$s_4 = -1,13905$	$s_5 = -0,519484$
$s_6 = -1,86479$	$s_7 = -1,18696$	$s_8 = -2,49246$	$s_9 = -1,78521$	$s_{10} = -3,06777$
$s_{11} = -2,34195$	$s_{12} = -3,60905$	$s_{13} = -2,87006$	$s_{14} = -4,12577$	$s_{15} = -3,37679$
$s_{16} = -4,62361$	$s_{17} = -3,86666$	$s_{18} = -5,10627$	$s_{19} = -4,34275$	$s_{20} = -5,57635$
$s_{21} = -4,80729$	$s_{22} = -6,03576$	$s_{23} = -5,26192$	$s_{24} = -6,48592$	$s_{25} = -5,70791$
$s_{26} = -6,92799$	$s_{27} = -6,14628$	$s_{28} = -7,36287$	$s_{29} = -6,57785$	$s_{30} = -7,7913$
$s_{31} = -7,00329$	$s_{32} = -8,21389$	$s_{33} = -7,42317$	$s_{34} = -8,63116$	$s_{35} = -7,83795$
$s_{36} = -9,04354$	$s_{37} = -8,24797$	$s_{38} = -9,45128$	$s_{39} = -8,65351$	$s_{40} = -9,85435$
$s_{41} = -9,05376$	$s_{42} = -10,2525$	$s_{43} = -9,44847$	$s_{44} = -10,6472$	$s_{45} = -9,85946$
$s_{46} = -11,0686$	$s_{47} = -10,3214$	$s_{48} = -11,5878$	$s_{49} = -10,9418$	$s_{50} = -12,2939$

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